INTERPLAY OF γ-RIGID AND γ-STABLE COLLECTIVE MOTION IN NEUTRON RICH RARE EARTH NUCLEI

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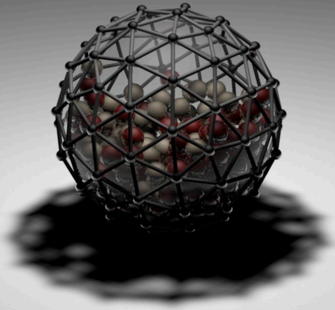

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The classical Hamiltonian function of the liquid drop model has 5 degrees of freedom, namely the two shape variables β and γ and the three Euler angles.

$$\mathcal{H} = \underbrace{\frac{B}{2}\left(\dot{\beta}^2 + \beta^2\dot{\gamma}^2\right)}_{T_{vib}} + \underbrace{\frac{1}{2}\sum_{k=1}^{3}\omega_k^2\mathcal{I}_k}_{T_{rot}} + V(\beta,\gamma).$$

Bohr-Mottelson Hamiltonian after quantization

Imposing a certain value for the γ shape variable, one reaches the γ-rigid version of the collective model which is interesting by itself due to its description of the basic rotation-vibration coupling.

- $\gamma \neq 0^\circ \Rightarrow$ 4 degrees of freedom $(\beta, \theta_1, \theta_2, \theta_3) \Rightarrow$ Davydov-Chaban Hamiltonian
  Davydov & Chaban NP **20** (1960) 499
- $\gamma = 0^\circ \Rightarrow$ 3 degrees of freedom $(\beta, \theta_1, \theta_2) \Rightarrow$ X(3)-type Hamiltonian
  Bonatsos *et. al.* PLB **632** (2006) 238

Although the γ-rigidity hypothesis is somewhat crude it provides simple approaches to the successful reproduction of the relevant experimental data.
Budaca EPJA **50** (2014) 87, PLB **739** (2014) 86; Buganu & Budaca PRC **91** (2015) 014306, JPG **42** (2015) 105106;

The similarity between the β excited bands of the X(5) and X(3) solutions addresses the question about the importance of rigidity in explaining the critical collective phenomena.

The kinetic energy operator $\hat{T}_{vib} + \hat{T}_{rot}$ in the five-dimensional shape phase space

$$T_s = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\sum_{k=1}^{3}\frac{Q_k^2}{\sin^2\left(\gamma - \frac{2}{3}\pi k\right)}\right]$$

In the prolate γ-rigid regime defined only by three degrees of freedom, the same operator gets a simpler form

$$T_r = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^2}\frac{\partial}{\partial\beta}\beta^2\frac{\partial}{\partial\beta} - \frac{\mathbf{Q}^2}{3\beta^2}\right]$$

The interplay between γ-stable and γ-rigid collective motion is achieved by considering the Hamiltonian:

$$H = \chi T_r + (1-\chi)T_s + V(\beta,\gamma), \quad 0 \leqslant \chi < 1$$

☞ **rigidity measure**

Budaca & Budaca JPG **42** (2015) 085103

β variable is separated from the γ-angular ones if the potential have the structure

$$v(\beta,\gamma) = \frac{2B}{\hbar^2}V(\beta,\gamma) = u(\beta) + (1-\chi)\frac{u(\gamma)}{\beta^2}$$

Factorizing the total wave function as $\Psi(\beta,\gamma,\Omega)=\xi(\beta)\varphi(\gamma,\Omega)$, the associated Schrödinger equation is separated in two parts:

γ-angular equation

$$\left[(1-\chi)\left(-\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma}+\sum_{k=1}^{3}\frac{Q_k^2}{4\sin^2\left(\gamma-\frac{2}{3}\pi k\right)}+u(\gamma)\right)+\frac{\chi}{3}\mathbf{Q}^2\right]\varphi(\gamma,\Omega)=W\varphi(\gamma,\Omega)$$

Small angle approximation $\Rightarrow$ $u(\gamma)=(3a)^2\frac{\gamma^2}{2}$

a - stiffness of γ oscillations

$\Downarrow$

$$W=3a(1-\chi)(n_\gamma+1)+\frac{L(L+1)-(1-\chi)K^2}{3},\quad \varphi(\gamma,\Omega)=\eta(\gamma)D^L_{MK}(\Omega)$$

$$\eta_{n_\gamma,|K|}(\gamma)=N_{n,|K|}\gamma^{|K/2|}\exp\left(-3a\frac{\gamma^2}{2}\right)L_n^{|K/2|}(3a\gamma^2),\quad n=\frac{1}{2}\left(n_\gamma-\left|\frac{K}{2}\right|\right)$$

β equation

$$\left[-\frac{\partial^2}{\partial\beta^2}-\frac{2(2-\chi)}{\beta}\frac{\partial}{\partial\beta}+\frac{W}{\beta^2}+u(\beta)\right]\xi(\beta)=\epsilon\xi(\beta)$$

Infinite square well (ISW) potential

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$$u(\beta)=\begin{cases}0, & \beta\leqslant\beta_W,\\ \infty, & \beta>\beta_W.\end{cases}$$

$\Downarrow$

$$\epsilon_{L,K,s,n_\gamma}(\beta_W)=\left(\frac{x_{s,\nu}}{\beta_W}\right)^2,\quad \xi_{L,K,s,n_\gamma}(\beta)=N_{s,\nu}\beta^{\chi-\frac{3}{2}}J_\nu\left(\frac{x_{s,\nu}\beta}{\beta_W}\right)$$

$$\nu=\left[\frac{L(L+1)-(1-\chi)K^2}{3}+\left(\frac{3}{2}-\chi\right)^2+(1-\chi)3a(n_\gamma+1)\right]^{\frac{1}{2}}$$

$x_{s,\nu}$ is s-th zero of the Bessel function $J_\nu(x_{s,\nu}\beta/\beta_W)$ and $n_\beta=s-1$.

Davidson (D) potential

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$$u(\beta)=\beta^2+\frac{\beta_0^4}{\beta^2}.$$

$\Rightarrow$

$$\epsilon_{LKn_\beta n_\gamma}=2n_\beta+p+\frac{5}{2},\quad \xi_{LKn_\beta n_\gamma}(\beta)=N_{n_\beta\nu}\beta^{p+\chi}e^{-\frac{\beta^2}{2}}L_n^{p+\frac{3}{2}}(\beta^2)$$

$$p=-\frac{3}{2}+\left[\frac{L(L+1)-(1-\chi)K^2}{3}+\left(\frac{3}{2}-\chi\right)^2+\beta_0^4+(1-\chi)3a(n_\gamma+1)\right]^{\frac{1}{2}}$$

The full solution after proper normalization and symmetrization reads:

$$\Psi_{LMKn_\beta n_\gamma}(\beta,\gamma,\Omega)=\xi_{L,K,n_\beta,n_\gamma}(\beta)\eta_{n_\gamma,|K|}(\gamma)\sqrt{\frac{2L+1}{16\pi^2(1+\delta_{K,0})}}\left[D^L_{MK}(\Omega)+(-)^LD^L_{M-K}(\Omega)\right]$$

The $B(E2)$ rates are calculated with the quadrupole transition operator $T^{(E2)}_\mu=t\beta q_\mu$

An identical β differential equation for determining the energy of the system is obtained if one starts from the classical picture of LDM:

$$\mathcal{H} = \frac{B}{2}\dot{\beta}^2 + (1-\chi)\frac{B}{2}\beta^2\dot{\gamma}^2 + (1-\chi)T_{rot}^{\gamma\neq 0} + \chi T_{rot}^{\gamma=0} + V(\beta,\gamma)$$

Due to its consistent geometrical construction, the LDM kinetic energy operator is given by a Laplacian in a generalized coordinate system $x_m = (\beta, \gamma, \Omega)$:

$$\hat{T} = -\frac{\hbar^2}{2}\nabla^2 = -\frac{\hbar^2}{2}\sum_{lm}\frac{1}{J}\frac{\partial}{\partial x^l}J\bar{g}^{lm}\frac{\partial}{\partial x^m},$$

where $J = \sqrt{det(g)}$ is the Jacobian of the transformation from the quadrupole coordinates

$$q_m(\beta,\gamma,\Omega) = \beta\left\{D^2_{m0}(\Omega)\cos\gamma + \frac{1}{\sqrt{2}}\left[D^2_{m2}(\Omega) + D^2_{m-2}(\Omega)\right]\sin\gamma\right\}.$$

to the curvilinear ones $\{x^l\}$ defined by the metric tensor:

$$g_{lm} = \sum_k \frac{\partial q_k}{\partial x^l}\frac{\partial q_k}{\partial x^m}, \quad \bar{g}^{lm} = \sum_k \frac{\partial x^k}{\partial q_l}\frac{\partial x^k}{\partial q_m}.$$

- Bohr-Mottelson model (5 variables)

$$g = B \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \beta^2 & 0 & 0 & 0 \\ 0 & 0 & 4\beta^2 \sin^2\left(\gamma - \frac{2\pi}{3}\right) & 0 & 0 \\ 0 & 0 & 0 & 4\beta^2 \sin^2\left(\gamma - \frac{4\pi}{3}\right) & 0 \\ 0 & 0 & 0 & 0 & 4\beta^2 \sin^2\gamma \end{pmatrix}$$

- Axial γ-rigid regime (3 variables)

$$g = B \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3\beta^2 & 0 \\ 0 & 0 & 3\beta^2 \sin^2\theta \end{pmatrix}$$

- Present case (5 variables)

$$\hat{T} = -\frac{\hbar^2}{2} \sum_{lm} \frac{1}{J} \frac{\partial}{\partial x^l} J \bar{G}^{lm} \frac{\partial}{\partial x^m},$$

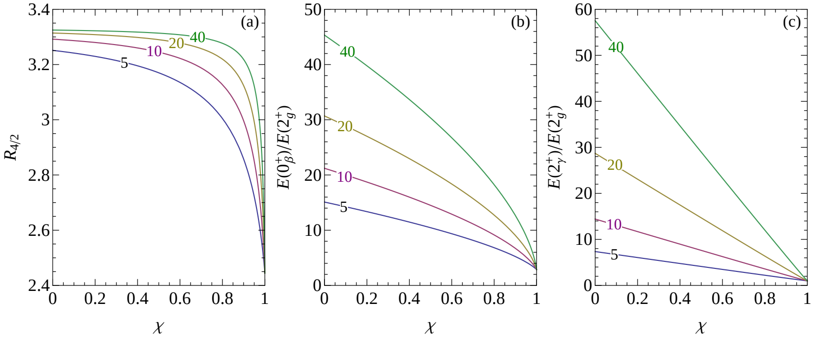
G_{lm} is a symmetric positive-definite bitensor not necessarily related to the metric g_{lm}.
Prochniak & Rohozinski JPG **36** (2009) 123101

Imbedding the χ dependence in G_{lm}, one obtains after quantization a differential equation whose eigenfunction differs from the previous one by a factor β^{χ}.

In order to have both model derivations equivalent

one must amend the integration measure in the quantum constructed model by the factor $\beta^{-2\chi}$. Budaca & Budaca EPJA **51** (2015) 126

The evolution as function of χ and a of theoretically evaluated (with ISW potential) spectral observables such as $R_{4/2} = E(4_g^+)/E(2_g^+)$ (a) ratio and the β (b) and γ (c) band heads normalized to the energy of the first excited state.

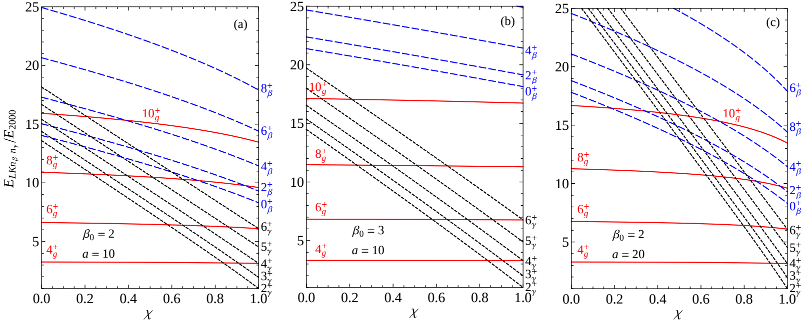


Bonatsos *et. al.* PLB **649** (2007) 394

$$\underset{\chi=0}{\text{ES-}X(5)} \xrightarrow{\ \chi\ } \underset{\chi=1}{X(3)}$$

Bonatsos *et. al.* PLB **632** (2006) 238

- **The low lying energy spectrum given as function of the rigidity parameter χ, for different values of the remaining parameters.**



Bonatsos *et. al.* PRC **76** (2007) 064312

$$\underset{\chi=0}{\text{ES-D}} \xrightarrow{\ \chi\ } \underset{\chi=1}{X(3)\text{-D}}$$

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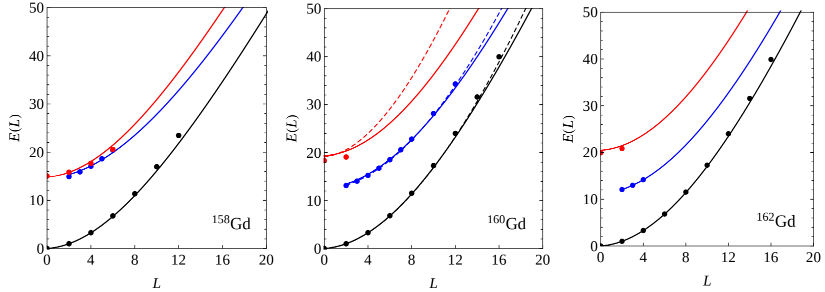
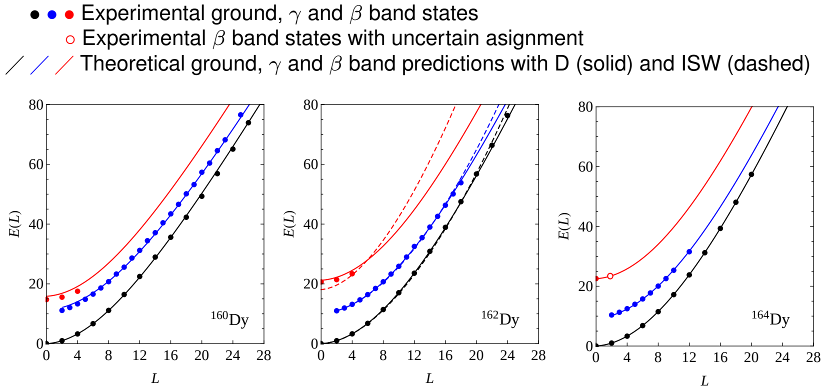
● ● ● Experimental ground, γ and β band states

/ / / Theoretical ground, γ and β band predictions with D (solid) and ISW (dashed)



χ	$3 \cdot 10^{-4}$	0.826 (0.948)	0.092
a	11.349	51.538 (168.899)	9.052
β_0	2.044	2.840	2.963
σ	0.601	0.768 (0.567)	0.574
States	14	18	12

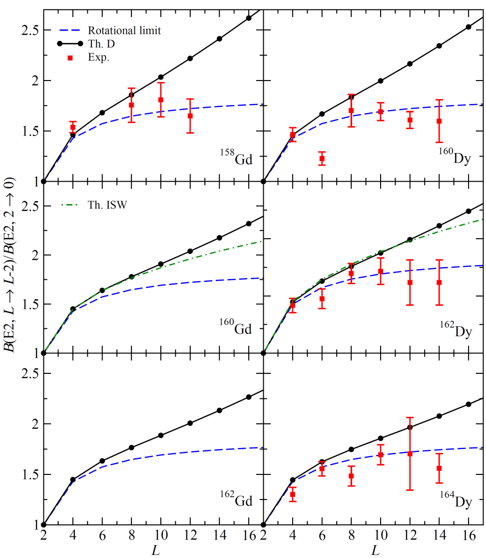
( ) - ISW fits



χ	0.423	0.067 (0.269)	0.734
a	14.519	7.934 (10.309)	24.473
β_0	2.442	3.056	3.205
σ	0.636	0.411 (0.845)	0.092
States	39	31	20

( ) - ISW fits

Theoretical g-g $E2$ transition probabilities compared with the available experimental data and with the rigid rotor predictions.



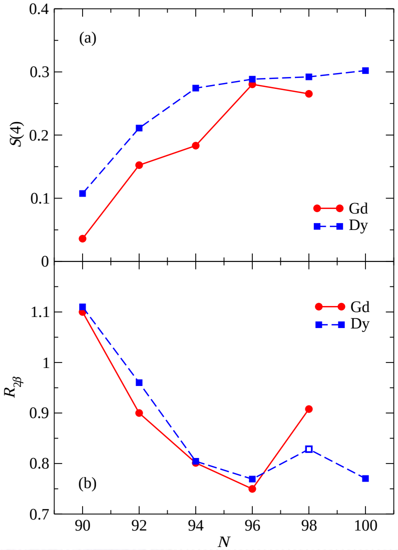
- Experimental points are situated between the rigid rotor and present model's predictions.
- The data are closer to the rigid rotor limit in case of the Dy isotopes.
- While the only measurements for the Gd isotopes associated to ^{158}Gd are closer to the present calculations.

Comparison of theoretical results with experiment and rigid rotor (R. R.) predictions for several interband $E2$ transition probabilities.

Nucleus	$\frac{2^+_\beta \to 2^+_g}{2^+_\beta \to 0^+_g}$	$\frac{2^+_\beta \to 4^+_g}{2^+_\beta \to 0^+_g}$	$\frac{2^+_\gamma \to 2^+_g}{2^+_\gamma \to 0^+_g}$	$\frac{2^+_\gamma \to 4^+_g}{2^+_\gamma \to 0^+_g}$
^{158}Gd	0.25(6)	4.48(75)	1.76(26)	0.079(14)
D	1.93	6.01	1.46	0.077
^{160}Gd			1.87(12)	0.189(29)
D	1.79	4.97	1.44	0.074
ISW	1.81	5.00	1.45	0.074
^{162}Gd				
D	1.76	4.80	1.44	0.074
^{160}Dy		2.52(44)	1.89(18)	0.133(14)
D	1.89	5.70	1.45	0.075
^{162}Dy			1.78(16)	0.137(12)
D	1.75	4.70	1.44	0.073
ISW	1.84	5.18	1.45	0.074
^{164}Dy			2.00(27)	0.240(33)
D	1.73	4.55	1.44	0.073
R.R.	1.43	2.57	1.43	0.071

- Experimental γ-g transitions rates are slightly underestimated.
- While β-g transitions rates are overestimated.
- Overall higher values than the R.R. predictions.
- Very weak dependence on the parameters for γ-g transitions.

Experimental evidences for the occurrence of ^{160}Gd and ^{162}Dy as singular points in their respective isotopic chains.



(a) γ band staggering

$$S(4) = \frac{E(4_\gamma) + E(2_\gamma) - 2E(3_\gamma)}{E(2_g^+)}$$

(b) Relative spacing of the lowest states in the β band

$$R_{2\beta} = \frac{E(2_\beta^+) - E(0_\beta^+)}{E(2_g^+)}$$

β equation for the spherical vibrator model with an energy dependent string constant:

$$\left[-\frac{\partial^2}{\partial\beta^2}+\frac{(\tau+1)(\tau+2)}{\beta^2}+k(\epsilon)\beta^2\right]f(\beta)=\epsilon f(\beta),\quad f(\beta)=\beta^2F(\beta)$$

$\Rightarrow$ quadratic equation for the energy of the system:

$$\epsilon=\sqrt{k(\epsilon)}\left(N+\frac{5}{2}\right),\quad N=2n_\beta+\tau$$

if the simplest energy dependence $k(\epsilon)=1+a\epsilon$ is chosen, which leads to

$$\epsilon_N = \left[\left(N+\frac{5}{2}\right)\frac{a}{2}+\sqrt{1+\left(N+\frac{5}{2}\right)^2\frac{a^2}{4}}\right]\left(N+\frac{5}{2}\right)$$

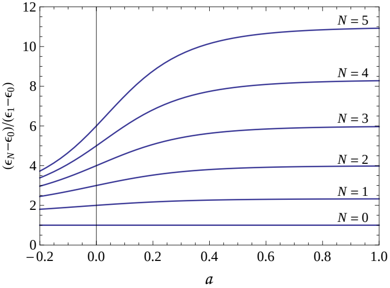
$$F_{n_\beta\tau}(\beta) = C_{n_\beta\tau}\left(\xi_{n_\beta\tau}\right)^\tau e^{-\frac{\left(\xi_{n_\beta\tau}\right)^2}{2}}L_{n_\beta}^{\tau+\frac{3}{2}}\left[\left(\xi_{n_\beta\tau}\right)^2\right],\quad \xi_{n_\beta\tau}=\sqrt{1+a\epsilon_{n_\beta\tau}}\beta$$

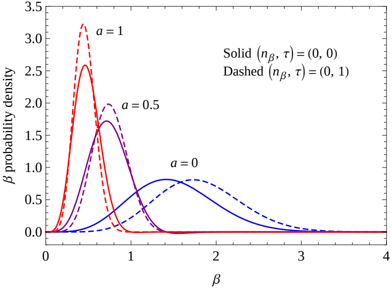
- Due to the energy dependence of the potential, the scalar product is modified as
Formanek, Lombard, Mares, Czech J. Phys. 54 (2004) 289

$$\beta^4 d\beta \quad \longrightarrow \quad \left[1 - \underbrace{\frac{\partial v(\beta,\epsilon)}{\partial \epsilon}}_{=a\beta^2} \right] \beta^4 d\beta,$$

in order to satisfy the continuity equation.

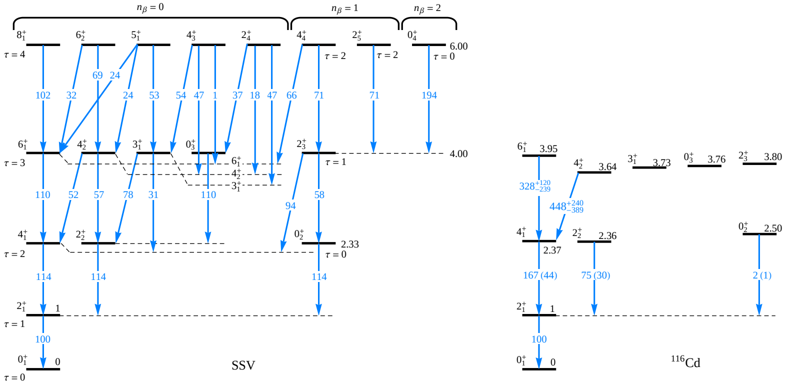
- Not a coherent theory for $a > 0$ because $\rho = \left[F_{n_\beta \tau}(\beta) \right]^2 \left(1 - a\beta^2\right) \beta^4$ is not positive definite.





Everything **OK** in the asymptotic limit of the a parameter, whose energy is described by a parameter free expression (except scale):

$$\epsilon_N = \left[\left(N+\frac{5}{2}\right)\frac{a}{2}+\sqrt{1+\left(N+\frac{5}{2}\right)^2\frac{a^2}{4}}\right]\left(N+\frac{5}{2}\right) \xrightarrow{a\gg\frac{4}{5}} \epsilon_N = \frac{a}{2}\left(N+\frac{5}{2}\right)^2$$



Other suitable experimental realizations: ^{104}Pd and ^{106}Pd.

Budaca PLB (2015)

CONCLUSIONS

- A simple exactly separable model was constructed by taking the kinetic energy of the Bohr Hamiltonian as a combination of prolate γ-rigid and γ-stable rotation-vibration kinetic operators.

- The relative weight of these two components is managed through a so called rigidity parameter which bridges the $X(3)$ and $X(3)$-D γ-rigid solutions to their γ-stable counterparts represented by ES-$X(5)$ and ES-D models when an ISW and respectively Davidson potential in β is adopted.

- The model was successfully applied for the description of the collective spectra for few heavier isotopes of Gd and Dy. In both cases a critical nucleus was identified through a discontinuous behavior in respect to the rigidity parameter and relevant experimental observables.

The proposed hybrid formalism unveils

alternative features of the collective motion in the Gd and Dy isotopic chains which are known to undergo shape phase transitions.

- Quantum starting point

$$\left[-\frac{\partial^2}{\partial\beta^2} - \frac{2(2-\chi)}{\beta}\frac{\partial}{\partial\beta} + \frac{W}{\beta^2} + u(\beta)\right]\xi(\beta) = \epsilon\xi(\beta)$$

$$\xi_{LKn_\beta n_\gamma}(\beta) = N_{n_\beta\nu}\beta^{p+\chi}e^{-\frac{\beta^2}{2}}L_n^{p+\frac{3}{2}}(\beta^2)$$

$$dV = \beta^{4-2\chi}\left|\sin 3\gamma\right| d\beta d\gamma d\Omega$$

- Classical starting point

$$\left[-\frac{\partial^2}{\partial\beta^2} - \frac{4}{\beta}\frac{\partial}{\partial\beta} + \frac{W}{\beta^2} + u(\beta)\right]\xi(\beta) = \epsilon\xi(\beta)$$

$$\xi_{LKn_\beta n_\gamma}(\beta) = N_{n_\beta\nu}\beta^{p}e^{-\frac{\beta^2}{2}}L_n^{p+\frac{3}{2}}(\beta^2)$$

$$dV = \beta^{4}\left|\sin 3\gamma\right| d\beta d\gamma d\Omega$$

$$\eta_{n_\gamma,|K|}(\gamma) = N_{n,|K|}\gamma^{|K/2|}\exp\left(-3a\frac{\gamma^2}{2}\right)L_n^{|K/2|}(3a\gamma^2), \quad n = \frac{1}{2}\left(n_\gamma - \left|\frac{K}{2}\right|\right)$$

$$W = 3\underbrace{a(1-\chi)}_{c}(n_\gamma+1) + \frac{L(L+1)-(1-\chi)K^2}{3}, \quad \varphi(\gamma,\Omega) = \eta(\gamma)D^L_{MK}(\Omega)$$

$$\Downarrow$$

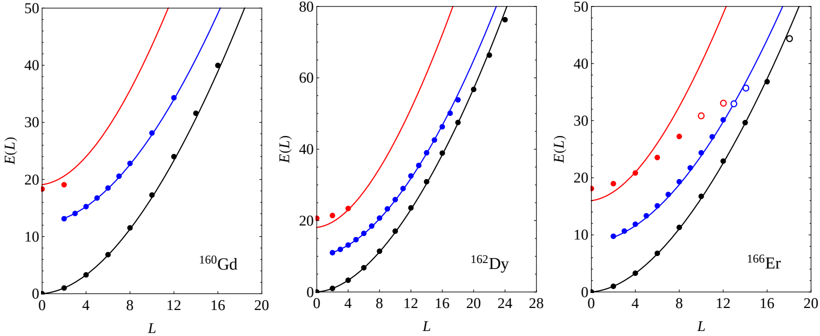
$$\eta_{n_\gamma,|K|}(\gamma) = N_{n,|K|}\gamma^{|K/2|}\exp\left(-\frac{3c}{1-\chi}\frac{\gamma^2}{2}\right)L_n^{|K/2|}\left(\frac{3c}{1-\chi}\gamma^2\right)$$

When $c = const.$ and $\chi \to 1$ the associated γ probability distribution acquires a strongly localized maximum at a value which tends asymptotically to zero.

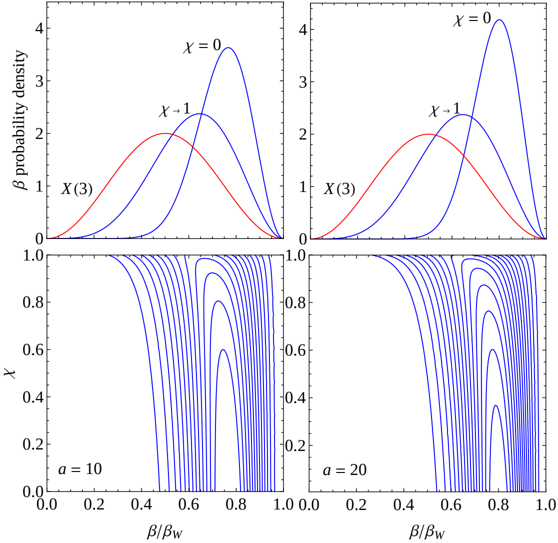
- Rigidity χ and γ stiffness a are interrelated.

$$\boxed{a \xrightarrow{\chi \to 1} \infty}$$

● ● ● Experimental ground, γ and β band states
○ ○ ○ Experimental ground, γ and β band states with uncertain asignment
/ / / Theoretical ground, γ and β band predictions



χ	0.948	0.269	0.848
a	168.899	10.309	41.191
σ	0.567	0.845	1.359
Nr. of states	19	32	24



- The ground state β probability density in respect to the $d\beta$ integration measure.