Shapes and Symmetries of Nuclei

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Shapes
short overview
Mathematician and statistician David George Kendall writes [*]:

In this paper ‘shape’ is used in the vulgar sense, and means what one would normally expect it to mean. [...] We here define ‘shape’ informally as ‘all the geometrical information that remains when location, scale [3] and rotational effects are filtered out from an object.’


[3] scaling = uniform scaling only.
Non-rigid shapes

Numerical Geometry of Non-Rigid Shapes by Alexander M. Bronstein, Michael M. Bronstein, Ron Kimmel
Two remarks on the problem: Non-rigid shape similarity. How to compare shapes that are susceptible to deformations?

1. One tries to look for quantitative measure of "distance" between two shapes.

2. In many cases, there exists a set of deformation invariants for a given body (deformation invariant similarity). For example: movements of fingers in the hand. The length of fingers remain constant (partial isometries)
These partial isometries are the deformation invariants. All deformation invariants define an intrinsic geometry of the shape.
Let $q_1, q_2$ and $q_3$ be curvilinear coordinates in $\mathbb{R}^3$.

General 'geometric shape'/surface equation in $\mathbb{R}^3$:

$$q_k = q_k(u, v) \text{ where } k = 1, 2, 3.$$

Assume $q_k \in L^2(S)$, where the compact subset $S \subset \mathbb{R}^2$.

Let the set $\{e_n(u, v)\}$ gives an orthonormal basis in $L^2(S)$ having an appropriate physical meaning then

$$q_k(u, v) = \sum_n \alpha_{n,k} e_n(u, v).$$
The expansion coefficients:

\[ \alpha_{n,k} = \int_S dudv \, \rho(u, v) \, e_n(u, v)^* q_k(u, v) \]

are new variables describing the nuclear surface in terms of this basis.

An example, multipole collective variables:

Coordinate space: \( \{q_1 = r, q_2 = \theta, q_3 = \phi\} \), where \( u = \theta, v = \phi \).
Surface: \( r = R(\theta, \phi) \in L^2(SO(2)) \).
Observables: \( \hat{A}_1 = \hat{J}^2, \hat{A}_2 = \hat{J}_z \). The basis \( \{e_n(u, v) = Y_{lm}(\theta, \phi)\} \):

\[ R(\theta, \phi) = R_0(1 + \sum_{\lambda, \mu} \alpha_{\lambda \mu}^{(l)ab} \star Y_{\lambda \mu}(\theta, \phi)). \]
Observables and shape parametrization 3/3

Physical meaning of the basis required for parametrization of surfaces one can achieve, for example, by:

- Introduce two commuting quantum observables \((A_1, A_2)\) (deformation invariants) defined in \(L^2(S)\). Construct the eigenvectors of these observables and use them as the required basis.

- Use the appropriate symmetries and basis of the corresponding irreducible representations.

- Construct by hand vectors with required properties.

- ????
Quantum shape – diffused shapes

Quantum shape = 'cloud' $\Leftrightarrow |\Psi(\alpha; x)|^2$

A few unsolved problems:
- How to define ”diffused shapes” $\approx$ ”quantum shapes”? $\mathcal{O} = \partial\{x : |\Psi(\alpha; x)|^2 \geq \epsilon\}$? How to parametrize such object? $\partial = \text{boundary of a set.}$
- What are invariants of nuclear deformations?
- What is ”distance” between nuclear shapes? Maybe a difference of some characteristic energies related to these shapes?

Some difficulties:
- The problem of ”monster shapes” – relation between geometric definition of the deformation parameters and ”shapes” related to quantum states.
- Center of mass coupling to deformation parameters.
- ...
Example: Problem of "the monster shapes"

An example, multipole collective variables:

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\[
R(\theta, \phi) = R_0 (1 + \sum_{\lambda, \mu} \alpha^{(lab)*}_{\lambda\mu} Y_{\lambda\mu}(\theta, \phi)).
\]

Has \( R(\theta, \phi) \) to be interpreted as real nuclear surface? The method requires (in this case) only a function on the 2D sphere.
$\alpha_{1\mu}$ as the shift of the quadrupole shape?
Figures: “Nuclear surfaces” 2/4

Monster quadrupole shapes!
Figures: “Nuclear surfaces” 3/4

Regular quadrupole shapes.
Regular quadrupole tetrahedral shape (left), strange quadrupole-octupole shape (right)
Hamiltonian and ‘shapes’

Nilsson Model, deformation by frequencies

\[ H = -\frac{\hbar^2}{2m} \Delta + \frac{m}{2} (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) \]

Geometric interpretation:

\[ \omega_i = \omega_0 \frac{r_0}{a_i}, \quad a_i = \text{ellipsoid axis}, \]
\[ \omega_x \omega_y \omega_z = \omega_0^3, \quad \text{volume conservation.} \]

‘dist’ deformed Hamiltonians

\[ H = -\frac{\hbar^2}{2m} \Delta + V(\text{dist}(\vec{r}, R(\alpha; \theta, \phi))) \]

Equipotential surfaces as shapes?
Hamiltonian and ’shapes’

**Shifted oscillator**

\[ H = T_{\text{coll}} + \frac{m}{2} \omega_0^2 (\xi - \xi_0)^2. \]

Geometric interpretation:

- For \( \xi_0 \neq 0 \) = deformed oscillator (what shape?).
- For \( \xi_0 = 0 \) = usually called “spherical”.
- However, usually \( \langle \psi | \xi^2 | \psi \rangle \neq 0 \) – dynamical deformation.
- The standard Bohr Hamiltonian do not contain any deformation parameter, the collective variable are at the same time deformation parameters. What is the corresponding shape:  
  a) this one obtained from the potential energy?  
  b) diffused ’shape’ obtained from the collective wave function?
Geometry versus Symmetry

Example: GCM+GOA
Rigid rotor: GCM+GOA 1/4

The generating function:

$$|\Omega\rangle = \hat{R}(\Omega)|\alpha\rangle$$

where $|\alpha\rangle$ – axially symmetric.

$H(\alpha)$ – axially symmetric intrinsic Hamiltonian (at the moment when both Laboratory and Intrinsic frames coincide)

At any other "moment" the Hamiltonian is

$$H' = \hat{R}(\Omega)H(\alpha)\hat{R}(\Omega)^\dagger$$

$$= \hat{R}(\Omega_1, \Omega_2, 0)H(\alpha)\hat{R}(\Omega_1, \Omega_2, 0)^\dagger$$

$\Omega_3 = 0$ because of axial symmetry.
Rigid rotor: GCM+GOA 2/4

The rotor Hamiltonian:

\[ H_{\text{rot}} = \frac{1}{2} \mathcal{J}^{-1}(\alpha) \hat{J}^2 + V(\alpha) \]

The eigenenergies:

\[ E_{\text{rot};J} = \frac{1}{2} \mathcal{J}(\alpha)^{-1} J(J + 1) + V(\alpha) \]

The eigenfunctions:

\[ r^J_{M, K=0}(\Omega) = \sqrt{2J + 1} D^J_{M0} (\Omega) \]

Here: the axial symmetry implies only \( K = 0 \).
The Hamiltonian \( H_{\text{rot}} \) is SO(3) invariant in respect to Lab. rotations.
Rigid rotor: GCM+GOA 3/4

The generating function:

\[ |\Omega\rangle = \hat{R}(\Omega)|\alpha\rangle \]

where \[ |\alpha\rangle = e^{-i\pi \hat{J}_k} |\alpha\rangle \]

\[ k = 1 \rightarrow x, \quad k = 2 \rightarrow y, \quad k = 3 \rightarrow z \]

and \[ |\alpha\rangle \] is \( D_2 \) symmetric.

\( H(\alpha) \) is \( D_2 \) symmetric intrinsic Hamiltonian (at the moment when both Laboratory and Intrinsic frames coincide).

At any other "moment" the Hamiltonian is

\[ H' = \hat{R}(\Omega) H(\alpha) \hat{R}(\Omega)^\dagger = \hat{R}(\Omega) H(\alpha) \hat{R}(\Omega)^\dagger \]
Rigid rotor: GCM+GOA 4/4

The rotor Hamiltonian:

\[ H_{\text{rot}} = \frac{1}{2} \sum_{k=1}^{3} \mathcal{J}_k^{-1}(\alpha)(\hat{J}_k)^2 + V(\alpha) \]

It can happen that: \( \mathcal{J}_x(\alpha)^{-1} = \mathcal{J}_y(\alpha)^{-1} \)

Than the eigenenergies:

\[ E_{\text{rot};JK} = \frac{1}{2} \left[ \frac{J(J + 1) - K^2}{\mathcal{J}_x(\alpha)} + \frac{K^2}{\mathcal{J}_z(\alpha)} \right] + V(\alpha) \]

And the eigenfunctions for \( K \neq 0 \):

\[ r_{M,K}^{(\pm)J}(\Omega) = \frac{1}{\sqrt{2}} \left[ r_{MK}^{J}(\Omega) \pm r_{MK}^{J}(\Omega) \right] \]

"Macroscopic" axial symmetry implies \( K \geq 0 \).
The Hamiltonian \( H_{\text{rot}} \) is \( SO(3) \) invariant in respect to Lab. rotations.
CONCLUSIONS - Shapes

- Artistic invention in creating ‘shapes’ in nuclear physics.
- Many very different parameters are called “deformation parameters”.
- Problem of interpretation of these parameters in terms of a distribution of nucleons in a nucleus i.e. shapes.
- Symmetries can be considered as ”deformation invariants”.
- Symmetry of the Hamiltonian and symmetry of quantum shape are often different.
Problems

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