Shapes and Symmetries of Nuclei

A. Góźdź
Institute of Physics, Dept. Math. Phys.,UMCS, Lublin, Poland

SDANCA 2015, Sofia

COLLABORATION

Artur Dobrowolski
IF UMCS, Lublin, Poland
Aleksandra Pędrak
IF UMCS, Lublin, Poland
Agnieszka Szulerecka
IF UMCS, Lublin, Poland
Jerzy Dudek
IPHC, Strasbourg, France
A. Gusev
JINR, Dubna, Russia
S. Vinitsky
JINR, Dubna, Russia

Shapes
short overview

Geometric shapes

Mathematician and statistician David George Kendall writes [*]:

> In this paper 'shape' is used in the vulgar sense, and means what one would normally expect it to mean. [...] We here define 'shape' informally as 'all the geometrical information that remains when location, scale [3] and rotational effects are filtered out from an object.'

[*] Kendall, D.G. (1984). "Shape Manifolds, Procrustean Metrics, and Complex Projective Spaces". Bulletin of the London Mathematical Society 16 (2): 81-121. doi:10.1112/blms/16.2.81.
[3] scaling = uniform scaling only.

Non-rigid shapes

Numerical Geometry of Non-Rigid Shapes by Alexander M. Bronstein, Michael M. Bronstein, Ron Kimmel
Two remarks on the problem: Non-rigid shape similarity. How to compare shapes that are susceptible to deformations?

1. One tries to look for quantitative measure of "distance" between two shapes.
2. In many cases, there exists a set of deformation invariants for a given body (deformation invariant similarity).
   For example: movements of fingers in the hand. The length of fingers remain constant (partial isometries)
   These partial isometries are the deformation invariants. All deformation invariants define an intrinsic geometry of the shape.

Observables and shape parametrization 1/3

Let q_1, q_2 and q_3 be curvelinear coordinates in R^3.

General 'geometric shape'/surface equation in $\mathbb{R}^3$:

$$q_k = q_k(u, v) \text{ where } k =, 1, 2, 3.$$

Assume $q_k \in L^2(S)$, where the compact subset $S \subset R^2$.

Let the set $\{e_n(u, v)\}$ gives an orthonormal basis in $L^2(S)$ having an appropriate physical meaning then

$$q_k(u, v) = \sum_n \alpha_{n,k}\, e_n(u, v).$$

Observables and shape parametrization 2/3

The expansion coefficients:

$$\alpha_{n,k} = \int_S dudv\, \rho(u,v)\, e_n(u,v)^\star q_k(u,v)$$

are new variables describing the nuclear surface in terms of this basis.

An example, multipole collective variables:

Coordinate space: $\{q_1 = r, q_2 = \theta, q_3 = \phi\}$, where $u = \theta, v = \phi$,
Surface: $r = R(\theta, \phi) \in L^2(SO(2))$.
Observables: $\hat{A}_1 = \hat{J}^2$, $\hat{A}_2 = \hat{J}_z$. The basis $\{e_n(u,v) = Y_{lm}(\theta,\phi)\}$:

$$R(\theta,\phi) = R_0(1 + \sum_{\lambda,\mu} \alpha^{(lab)\star}_{\lambda\mu} Y_{\lambda\mu}(\theta,\phi)).$$

Observables and shape parametrization 3/3

Physical meaning of the basis required for parametrization of surfaces one can achive, for example, by:

- Introduce two commuting quantum observables (A_1, A_2) (deformation invariants) defined in $L^2(S)$. Construct the eigenvectors of these obervables and use them as the required basis.
- Use the appropriate symmetries and basis of the corresponding irreducible representations.
- Construct by hand vectors with required properties.
- ????

Quantum shape – diffused shapes

Quantum shape = 'cloud' $\Leftarrow |\Psi(\alpha; x)|^2$

A few unsolved problems:

- How to define "diffused shapes" $\approx$ "quantum shapes"?
  $\mathcal{O} = \partial\{x : |\Psi(\alpha; x)|^2 \geq \epsilon\}$? How to parametrize such object?
  ∂ = boundary of a set.
- What are invariants of nuclear deformations?
- What is "distance" between nuclear shapes? Maybe a difference of some characteristic energies related to these shapes?

Some difficulties:

- The problem of "monster shapes" – relation between geometric definition of the deformation parameters and "shapes" related to quantum states.
- Center of mass coupling to deformation parameters.
- ....

Example: Problem of "the monster shapes"

An example, multipole collective variables:

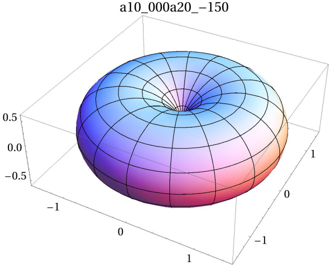
Coordinate space: $\{q_1 = r, q_2 = \theta, q_3 = \phi\}$, where $u = \theta, v = \phi$,
Surface: $r = R(\theta, \phi) \in L^2(SO(2))$.
Observables: $\hat{A}_1 = \hat{J}^2$, $\hat{A}_2 = \hat{J}_z$. The basis $\{e_n(u, v) = Y_{lm}(\theta, \phi)\}$:

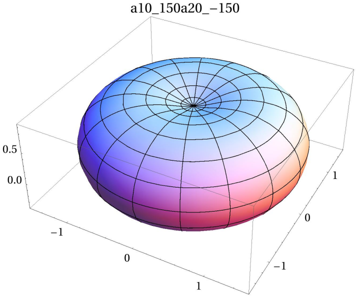
$$R(\theta, \phi) = R_0(1 + \sum_{\lambda,\mu} \alpha^{(lab)\star}_{\lambda\mu} Y_{\lambda\mu}(\theta, \phi)).$$

Has $R(\theta, \phi)$ to be interpreted as real nuclear surface? The method requires (in this case) only a function on the 2D sphere.

Figures: "Nuclear surfaces" 1/4

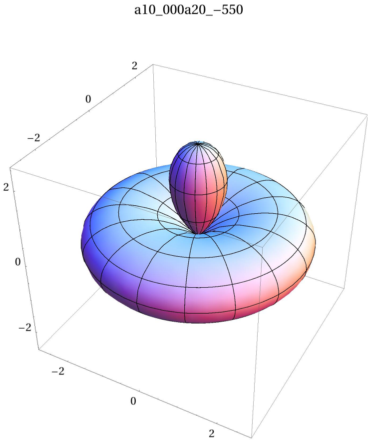
$\alpha_{1\mu}$ as the shift of the quadrupole shape ?

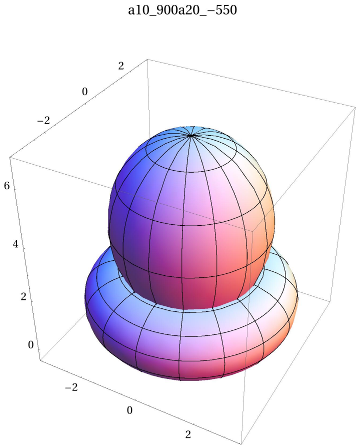




Figures: "Nuclear surfaces" 2/4

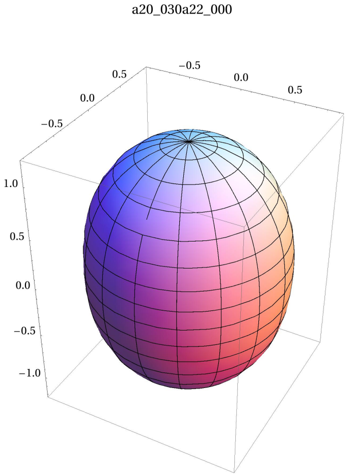
Monster quadrupole shapes !

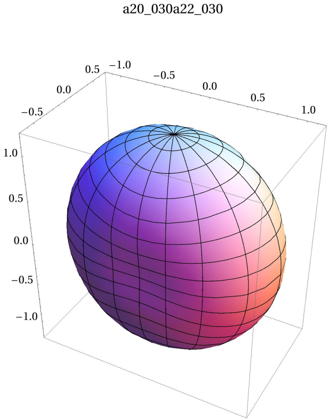




Figures: "Nuclear surfaces" 3/4

Regular quadrupole shapes.

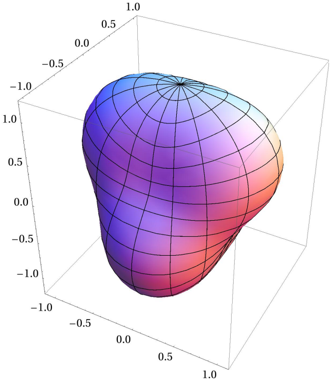




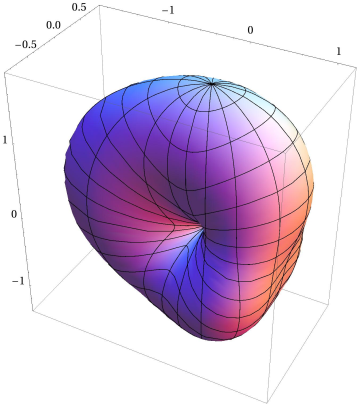
Figures: “Nuclear surfaces” 4/4

Regular quadrupole tetrahedral shape (left), strange quadrupole-octupole shape (right)

a20_030a22_005a32_030



a20_090a22_090a30_030a32_030a33_05



Hamiltonian and 'shapes'

Nillson Model, deformation by frequences

$$H = \frac{-\hbar^2}{2m}\triangle + \frac{m}{2}\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right)$$

Geometric interpretation:

$$\omega_i = \omega_0 \frac{r_0}{a_i},\ \ a_i = \text{ellipsoid axis},$$

$$\omega_x \omega_y \omega_z = \omega_0^3,\ \text{volume conservation.}$$

'dist' deformed Hamiltonians

$$H = \frac{-\hbar^2}{2m}\triangle + V(\mathrm{dist}(\vec{r}, R(\alpha; \theta, \phi))$$

Equipotential surfaces as shapes?

Hamiltonian and 'shapes'

Shifted oscillator

$$H = T_{coll} + \frac{m}{2}\omega_0^2(\xi - \xi_0)^2.$$

Geometric interpretation:

- For $\xi_0 \neq 0$ = deformed oscillator (what shape?).
- For $\xi_0 = 0$ = usually called "spherical".
- However, usually $\langle\psi|\xi^2|\psi\rangle \neq 0$ – dynamical deformation.
- The standard Bohr Hamiltonian do not contain any deformation parameter, the collective variable are at the same time deformation parameters. What is the corresponding shape:
  a) this one obtained from the potential energy?
  b) diffused 'shape' obtained from the collective wave function?

Geometry versus Symmetry

Example: GCM+GOA

Rigid rotor: GCM+GOA 1/4

The generating function:

$$|\Omega\rangle = \hat{R}(\Omega)|\alpha\rangle$$

where $|\alpha\rangle$ – axially symmetric.

$H(\alpha)$ – axially symmetric intrinsic Hamiltonian (at the moment when both Laboratory and Intrinsic frames coincide)
At any other "moment" the Hamiltonian is

$$H' = \hat{R}(\Omega)H(\alpha)\hat{R}(\Omega)^\dagger = \hat{R}(\Omega_1, \Omega_2, 0)H(\alpha)\hat{R}(\Omega_1, \Omega_2, 0)^\dagger$$

$\Omega_3 = 0$ because of axial symmetry.

Rigid rotor: GCM+GOA 2/4

The rotor Hamiltonian:

$$H_{rot} = \frac{1}{2}\mathcal{J}^{-1}(\alpha)\hat{J}^2 + V(\alpha)$$

The eigenenergies:

$$E_{rot;J} = \frac{1}{2}\mathcal{J}(\alpha)^{-1}\, J(J+1) + V(\alpha)$$

The eigenfunctions:

$$r^J_{M,K=0}(\Omega) = \sqrt{2J+1}D^{J\star}_{M0}(\Omega)$$

Here: the axial symmetry implies only $K = 0$.
The Hamiltonian H_{rot} is $\mathrm{SO}(3)$ invariant in respect to Lab. rotations.

Rigid rotor: GCM+GOA 3/4

The generating function:

$$|\Omega\rangle = \hat{R}(\Omega)|\alpha\rangle$$

where $|\alpha\rangle = e^{-i\pi \hat{J}_k}|\alpha\rangle$
$k = 1 \rightarrow x, k = 2 \rightarrow y, k = 3 \rightarrow z$ and $|\alpha\rangle$ – D_2 symmetric.

$H(\alpha)$ – D_2 symmetric intrinsic Hamiltonian (at the moment when both Laboratory and Intrinsic frames coincide)
At any other "moment" the Hamiltonian is

$$H' = \hat{R}(\Omega)H(\alpha)\hat{R}(\Omega)^\dagger = \hat{R}(\Omega)H(\alpha)\hat{R}(\Omega)^\dagger$$

Rigid rotor: GCM+GOA 4/4

The rotor Hamiltonian:

$$H_{rot} = \frac{1}{2}\sum_{k=1}^{3} \mathcal{J}_k^{-1}(\alpha)(\hat{J}_k)^2 + V(\alpha)$$

It can happens that: $\mathcal{J}_x(\alpha)^{-1} = \mathcal{J}_y(\alpha)^{-1}$
Than the eigenenergies:

$$E_{rot;JK} = \frac{1}{2}\left[\frac{J(J+1)-K^2}{\mathcal{J}_x(\alpha)} + \frac{K^2}{\mathcal{J}_z(\alpha)}\right] + V(\alpha)$$

And the eigenfunctions for $K \neq 0$:

$$r^{(\pm)J}_{M,K}(\Omega) = \frac{1}{\sqrt{2}}\left[r^J_{MK}(\Omega) \pm r^J_{M,-K}(\Omega)\right]$$

"Macroscopic" axial symmetry implies $K \geq 0$.
The Hamiltonian H_{rot} is $\mathrm{SO}(3)$ invariant in respect to Lab. rotations.

CONCLUSIONS - Shapes

- Artistic invention in creating 'shapes' in nuclear physics.
- Many very different parameters are called “deformation parameters”.
- Problem of interpretation of these parameters in terms of a distribution of nucleons in a nucleus i.e. shapes.
- Symmetries can be considered as ”deformation invariants”.
- Symmetry of the Hamiltonian and symmetry of quantum shape are often different.

Problems

???

?????????

?????????????????

???????????????????????????

????????????????????????????????????

??