Exact solutions of the extended pairing interactions in Bose and Fermi Many-Body systems

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1. E_2 algebraic structure

Let b, b^+ be annihilation and creation operator of bosons satisfying $[b, b^+] = 1, [b^+ b, b] = -b$, $[b^+ b, b^+] = b^+$. One can introduce functionals

$$\tilde{b} = f(b^+ b) b, \qquad \tilde{b}^+ = b^+ f(b^+ b),$$

$$f(b^+ b) = 1 / \sqrt{b^+ b + 1}$$

Then, the new algebra $\{\tilde{b}, \tilde{b}^+, n = b^+ b\}$ satisfies

$$[n, \tilde{b}] = -\tilde{b}, \ [n, \tilde{b}^+] = b^+$$

$$[\tilde{b}, \tilde{b}^+] = \delta_{n0}$$

Similarly, one can also do the same thing for fermion pairing algebras:

Let $N_j = \sum_m a^\dagger_{jm} a_{jm}$ be the number operator of fermions in a single-j shell and

$S^\dagger_j = \sum_{m>0} (-1)^{j-m} a^\dagger_{jm} a^\dagger_{j-m}$ be the monopole pair creation operator in the j-orbit. One can fermionic E_2 algebraic with

$$\tilde{S}^\dagger_j = \frac{1}{\sqrt{(Q_j - \hat{S}^0_j + 1)(Q_j + \hat{S}^0_j)}} S^\dagger_j$$

where $Q_j = \frac{1}{2}(\Omega_j - \nu_j)$ and $\hat{S}^0_j = \frac{1}{2}(N_j - \Omega_j)$ are quasi-spin and its third component in the j-orbit.

Let $|0\rangle$ be the vacuum state satisfying $a_{jm}|0\rangle = 0 \; \forall \; j$,
For the seniority $\nu_j \neq 0$ case, the lowest weight state for the quasi-spin Q_j is denoted as $|\nu_j \rho_j\rangle$ satisfying
$S_j|\nu_j\rho_j\rangle = 0 \qquad |Q_j, -Q_j; \nu_j\rho_j\rangle \equiv |\nu_j\rho_j\rangle$,

$$|Q_j, -Q_j + n; \nu_j\rho_j\rangle = \tilde{S}_j^{\dagger n}|\nu_j\rho_j\rangle$$

with $n = 0, 1, \cdots, 2Q_j$, which satisfies

$$\begin{pmatrix} S_j^\dagger S_j \\ \\ S_j^0 \end{pmatrix} |Q_j, -Q_j + n; \nu_j\rho_j\rangle = \begin{pmatrix} n(Q_j - n + 1) \\ \\ -Q_j + n \end{pmatrix} |Q_j, -Q_j + n; \nu_j\rho_j\rangle.$$

A set of operators $\{\tilde{S}_{j_i},\ \tilde{S}^\dagger_{j_i},\ N_{j_i}\}$ $(i = 1, 2, \cdots)$ **satisfy the commutation relations**

$$[N_{j'}/2,\ \tilde{S}_j] = -\delta_{jj'}\tilde{S}_j,\ [N_{j'}/2,\ \tilde{S}^\dagger_j] = \delta_{jj'}\tilde{S}^\dagger_j,\ [\tilde{S}_{j'},\ \tilde{S}^\dagger_j] = \delta_{jj'}\delta_{N_j 0}.$$

2. Applications to Boson models

(1) The extended Bose-Hubbad model

Similar to the original BH model, we consider an extended BH model with number-dependent multi-site hoping terms[Int J Theor Phys (2015) 54:2204]

$$\hat{H} = -t_0 \sum_{k=1}^{\infty} \sum_{j_1 \leq \cdots \leq j_k} \tilde{b}^\dagger_{j_1} \cdots \tilde{b}^\dagger_{j_k} \sum_{j'_1 \leq \cdots \leq j'_k} \tilde{b}_{j'_1} \cdots \tilde{b}_{j'_k} + \sum_j V(\hat{n}_j) + \sum_j \epsilon_j \hat{n}_j,$$

two groups of site-indices $\{j_1, \cdots, j_k\}$ and $\{j'_1, \cdots, j'_k\}$ with the restriction that no any one of $\{j_1, \cdots, j_k\}$ equals to any one of $\{j'_1, \cdots, j'_k\}$.

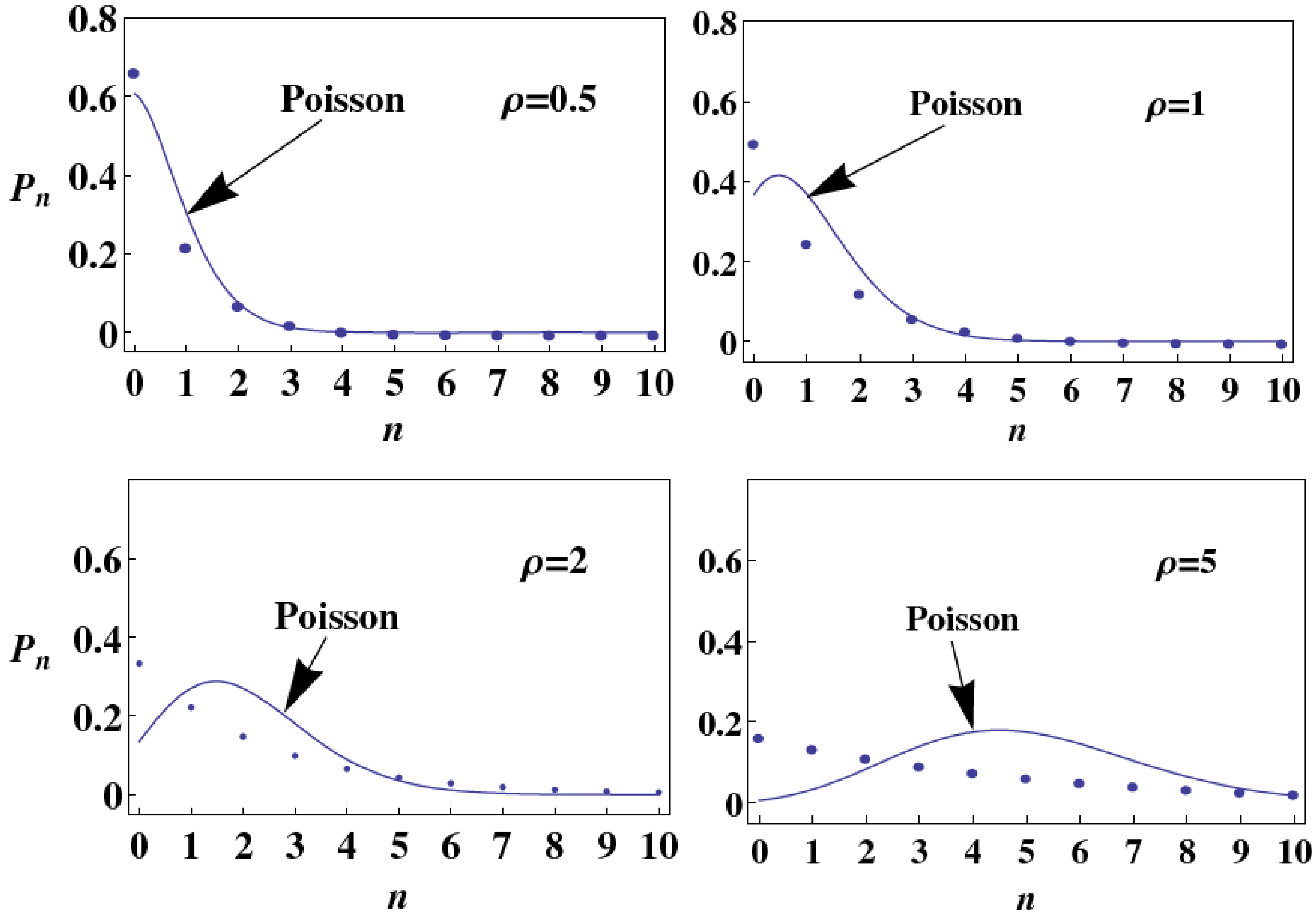
(a) The superfluid phase with $V(\hat{n}_j) = 0$ and $\epsilon_j = 0 \ \forall \ j$. The ground state in this case is non-degenerate and mostly delocalized with

$$|N\rangle_{\mathrm{g}} = \mathcal{N} \sum_{n_1,\cdots,n_M} |n_1, \cdots, n_M\rangle \qquad \mathcal{N}^{-2} = D(N, M).$$

The corresponding ground state energy is given by

$$E_{\mathrm{g}}^{\mathrm{SF}} = -t_0(D(N, M) - 1)$$

While other excited states are all degenerate with excitation energy being zero like the BCS spectrum.



The ground state probability to detect n particles on a given lattice site with $M = 10^5$ sites

$$P_n = \sum_{n_{i\neq 1}} |\langle n_1 = n, \{n_{i\neq 1}\}|N\rangle_{\mathrm{g}}|^2 \qquad \rho = N/M$$

(b) General solutions

exactly solvable when ϵ_j ($j = 1, \cdots, M$) are not equal one another

$$|N, \zeta\rangle = \sum_{n_1, \cdots, n_M} C^{(\zeta)}_{n_1, \cdots, n_M} |n_1, \cdots, n_M\rangle,$$

where the sum is restricted with $\sum_{j=1}^{M} n_j = N$,

$$C^{(\zeta)}_{n_1, \cdots, n_M} = \frac{1}{F(n_1, \cdots, n_M)},$$

$$F(n_1, \cdots, n_M) = E^{(\zeta)} - t_0 - \sum_j V(n_j) - \sum_{j=1}^{M} \epsilon_j n_j$$

$$- t_0 \sum_{n_1, \cdots, n_M} \frac{1}{F(n_1, \cdots, n_M)} = 1.$$

When ϵ_j $(j = 1, \cdots, M)$ are not equal one another, let $x_\mu = t_0 + \sum_{j=1}^{M} V(n_j) + \sum_{j=1}^{M} \epsilon_j n_j$ with $x_1 < x_2 < x_3 < \cdots < x_{D(N,M)}$, zeros of the polynomial $E^{(\zeta)}$, satisfy either the interlacing condition

$$x_1 < E^{(1)} < x_2 < E^{(2)} < x_3 < \cdots \text{ or}$$

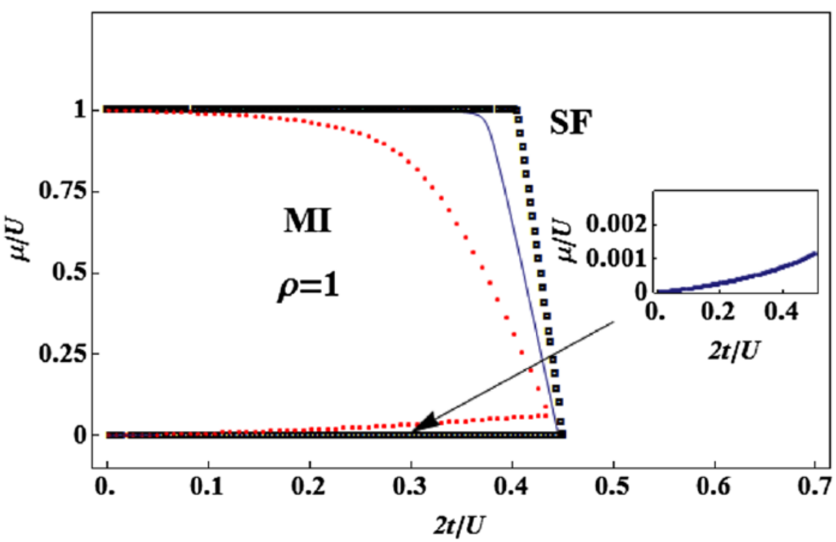
$$-\infty < x_1 < E^{(1)} < x_2 < E^{(2)} < x_3 < \cdots$$

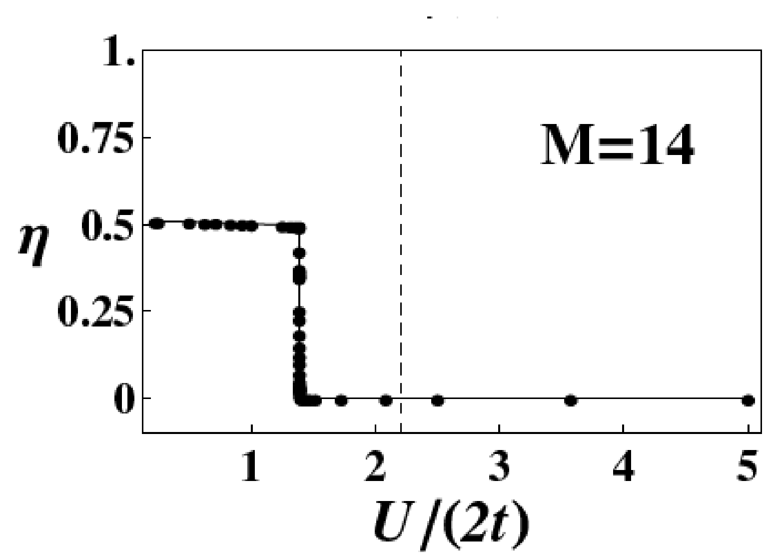
They are with in the open intervals $(-\infty, x_1), (x_1, x_2), \cdots,$ $(x_{D(N,M)-1}, x_{D(N,M)})$ or within $(x_1, x_2), (x_2, x_3), \cdots, (x_{D(N,M)}, +\infty)$.

(c) Special cases [exact ground state]

Table 1 The degree of the polynomial given by (14) for the integer filling cases with $\epsilon_j = 0 \; \forall \; j$ and $\rho = N/M = 1$ and compared with the dimension of the Hilbert subspace in full matrix diagonalization for the Hamiltonian (1) for some specific $N = M$ cases

	Degree of the polynomial	$D(N, M)$
$N = M = 6$	9	462
$N = M = 10$	27	92378
$N = M = 12$	39	1352078
$N = M = 14$	54	20058300





(2) Application to the IBM

Using the $SU_d(1,1)$ and $SU_s(1,1)$ generators, we can build an extended IBM Hamiltonian (EXT) as

$$\hat{H}_c = \Delta \hat{n}_d + \frac{\lambda}{N}(S_s^+ S_s^- + S_d^+ S_d^-) - g_2 \sum_{k=1}^{\infty} (\tilde{S}_s^{+k} \tilde{S}_d^{-k} + \tilde{S}_d^{+k} \tilde{S}_s^{-k}),$$

$$\tilde{S}_d^+ = S_d^+ \frac{1}{\sqrt{(S_d + S_d^0)(S_d^0 - S_d + 1)}}, \quad \tilde{S}_d^- = (\tilde{S}_d^+)^\dagger,$$

$$\tilde{S}_s^+ = S_s^+ \frac{1}{\sqrt{(S_s + S_s^0)(S_s^0 - S_s + 1)}}, \quad \tilde{S}_s^- = (\tilde{S}_s^+)^\dagger.$$

where $\Delta = \epsilon_d - \epsilon_s > 0$ is the energy gap of s and d bosons, and $\lambda > 0$ and $g_2 > 0$ are real parameters.[Phys. Rev. C91 (2015) 034305]

the $\tilde{E}_2$ algebra ⟹ $[\tilde{S}_\sigma^+, \tilde{S}_\rho^-] = -\delta_{\rho\sigma}\delta_{S_\rho S_\rho^0}, \ [S_\sigma^0, \tilde{S}_\rho^\pm] = \pm\delta_{\sigma\rho}S_\rho^\pm.$

$$|N,\zeta,\tau\alpha L\rangle = \sum_{\xi=0}^{\frac{1}{2}(N-\tau-\tau_s)} C_\xi^{(\zeta)}|\xi\rangle, \qquad |\xi\rangle \equiv |\tau_s;\xi;\tau\alpha LM\rangle \qquad C_\xi^{(\zeta)} = \frac{1}{F^{(\zeta)}(\xi)},$$

$$F^{(\zeta)}(\xi) = E_{\tau,L}^{(\zeta)} - g_2 - \frac{\lambda}{2N}\xi(2\tau+2\xi+3)$$
$$- \frac{\lambda}{4N}(N-\tau-2\xi)(N-\tau-2\xi-1) - \Delta(\tau+2\xi),$$

ζ,τ	E(5)	EXT		CQ		IBM fit [15]
		$N=60$	$N=1000$	$N=60$	$N=1000$	$N=60$
1,0	0.00	0.00	0.00	0.00	0.00	0.00
1,1	1.00	1.00	1.00	1.00	1.00	1.00
1,2	2.20	2.14	2.18	2.12	2.10	2.21
2,0	3.03	3.59	3.50	2.60	2.45	3.05
1,3	3.59	3.45	3.55	3.33	3.29	3.61
2,1	4.80	5.16	5.23	3.91	3.71	4.51
1,4	5.17	4.93	5.13	4.64	4.56	5.16
2,2	6.78	6.95	7.19	5.30	5.03	6.11
3,0	7.58	8.02	8.31	5.68	5.30	6.68
1,5	6.93	6.59	6.91	6.02	5.89	6.85
1,6	8.88	8.44	8.91	7.46	7.28	–
2,3	8.97	8.95	9.37	6.76	6.41	8.67
3,1	10.11	10.39	10.90	7.24	6.75	8.51
2,4	11.36	11.15	11.75	8.27	7.85	–
3,2	12.85	12.96	13.71	8.85	8.25	10.44
4,0	13.64	13.98	14.79	9.16	8.48	–
2,5	13.95	13.54	14.36	9.85	9.34	–
3,3	15.81	15.74	16.73	10.50	9.79	–
2,6	16.73	16.13	17.18	11.48	10.88	–
4,1	16.93	17.13	18.23	10.92	10.09	–
	χ^2	0.09	0.17	0.75	0.86	0.40

Comparison of low-lying level energies and relevant $B(E2)$ values of ^{102}Pd, ^{134}Ba, and ^{128}Xe

	E(5)	^{102}Pd	EXT	SET	^{134}Ba	EXT	CQ_5	^{128}Xe	EXT	CQ_6
0_1^+	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2_1^+	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2_2^+	2.20	2.76	2.21	2.58	1.93	2.31	2.15	2.29	2.14	2.15
4_1^+	2.20	2.29	2.21	2.58	2.32	2.31	2.15	2.44	2.14	2.15
0_ζ^+	3.03	2.98	2.69	2.92	3.57	2.92	3.18	3.74	3.38	3.10
0_τ^+	3.59	2.87	3.23	3.70	2.91	3.31	3.43	4.44	3.52	3.42
3_1^+	3.59	3.79	3.23	3.70	2.72	3.31	3.43	3.38	3.52	3.42
4_2^+	3.59	3.84	3.23	3.70	3.26	3.31	3.43	3.79	3.52	3.42
6_1^+	3.59	3.72	3.23	3.70	3.66	3.31	3.43	4.11	3.52	3.42
2_3^+	4.80	3.49	3.72	4.14	3.36	3.92	4.70	4.73	5.11	4.59
$B(E2:4_1^+ \to 2_1^+)$	167.4	154.5	143.9	187.8	154.8	141.9	140.8	146.7	151.3	147.2
$B(E2:6_1^+ \to 4_1^+)$	216.9	–	137.5	257.6	–	129.6	143.8	181.6	172.5	161.7
$B(E2:2_2^+ \to 2_1^+)$	167.4	45.0	143.9	187.8	217.2	141.9	140.8	119.4	151.3	147.2
$B(E2:2_2^+ \to 0_1^+)$	0.0	6.0	0.0	0.0	1.25	0.0	0.0	1.6	0.0	0.0
$B(E2:4_2^+ \to 2_1^+)$	0.0	9.0	0.0	0.0	–	0.0	0.0	–	0.0	0.0
$B(E2:4_2^+ \to 2_2^+)$	113.6	136.4	72.0	136.4	–	67.9	75.3	–	90.4	84.7
$B(E2:4_2^+ \to 4_1^+)$	103.3	<24.0	65.5	121.2	–	61.7	68.5	–	82.1	77.0
$B(E2:3_1^+ \to 2_2^+)$	154.9	–	98.2	184.8	12.8	92.5	102.7	–	123.2	115.5
$B(E2:3_1^+ \to 4_1^+)$	62.0	–	39.3	72.7	–	37.0	41.1	–	49.3	46.2
$B(E2:0_\tau^+ \to 2_2^+)$	216.9	291.0	137.5	257.6	–	129.6	143.8	–	172.5	161.7
$B(E2:0_\tau^+ \to 2_1^+)$	0.0	<0.001	0.0	0.0	3.0	0.0	0.0	–	0.0	0.0
$B(E2:2_3^+ \to 0_\zeta^+)$	72.2	–	47.2	109.1	–	43.7	56.6	–	66.8	64.6
$B(E2:0_\zeta^+ \to 2_1^+)$	86.8	39.4	68.5	163.6	41.7	57.3	64.5	–	93.4	72.3

For ^{104}Ru, ^{108}Pd, and ^{114,116}Cd.

	E(5)	^{104}Ru	EXT	SET	^{108}Pd	EXT	SET	^{116}Cd	EXT	SET	CQ_8	^{114}Cd	EXT	SET	CQ_9
0_1^+	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2_1^+	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2_2^+	2.20	2.49	2.03	2.22	2.14	2.03	2.20	2.36	2.07	2.66	2.14	2.17	2.00	2.00	2.14
4_1^+	2.20	2.48	2.03	2.22	2.42	2.03	2.20	2.37	2.07	2.66	2.14	2.30	2.00	2.00	2.14
0_ζ^+	3.03	2.76	3.04	2.34	2.43	2.41	2.30	2.69	2.43	3.09	3.00	2.03	2.00	2.00	2.96
0_τ^+	3.59	3.73	3.25	3.27	3.03	3.21	3.23	2.50	3.22	3.80	3.41	2.33	3.00	3.00	3.40
3_1^+	3.59	3.47	3.25	3.27	3.08	3.13	3.23	3.73	3.22	3.80	3.41	3.34	3.00	3.00	3.40
4_2^+	3.59	4.20	3.25	3.27	3.75	3.13	3.23	3.98	3.22	3.80	3.41	3.10	3.00	3.00	3.40
6_1^+	3.59	4.35	3.25	3.27	4.08	3.13	3.23	3.95	3.22	3.80	3.41	3.56	3.00	3.00	3.40
2_3^+	4.80	4.23	4.41	3.66	3.32	3.60	4.36	3.20	3.72	4.41	4.45	2.44	3.00	–	4.40
$B(E2: 4_1^+ \to 2_1^+)$	167.4	134.5	154.3	195.2	147.5	163.5	–	163.9	175.0	188.1	154.7	199.4	177.8	–	157.2
$B(E2: 6_1^+ \to 4_1^+)$	216.9	190.4	184.8	276.8	216.2	202.1	–	327.8	215.0	250.7	181.9	382.6	233.3	–	188.7
$B(E2: 2_2^+ \to 2_1^+)$	167.4	67.3	154.3	195.2	143.4	163.5	–	74.5	175.0	188.1	154.7	70.7	177.8	–	157.2
$B(E2: 2_2^+ \to 0_1^+)$	0.0	4.8	0.0	0.0	0.01	0.00	0.0	–	0.0	0.0	0.0	–	0.0	0.0	0.0
$B(E2: 4_2^+ \to 2_1^+)$	0.0	–	0.0	0.0	–	0.0	0.0	–	0.0	0.0	0.0	1.6	0.0	0.0	0.0
$B(E2: 4_2^+ \to 2_2^+)$	113.6	82.7	96.8	145.2	111.1	105.7	–	–	117.9	131.3	95.3	102.9	122.2	–	98.8
$B(E2: 4_2^+ \to 4_1^+)$	103.3	45.2	88.0	132.1	60.6	96.2	–	–	107.1	119.4	86.6	54.7	111.1	–	89.9
$B(E2: 3_1^+ \to 2_2^+)$	154.9	126.8	132.0	198.2	–	144.4	–	181.8	160.7	179.1	129.9	–	166.7	–	134.8
$B(E2: 3_1^+ \to 4_1^+)$	62.0	27.4	52.8	79.2	–	57.7	–	53.6	62.1	71.6	52.0	–	66.7	–	53.9
$B(E2: 0_\tau^+ \to 2_2^+)$	216.9	–	184.8	276.8	<36.4	202.1	–	–	225.0	250.7	181.9	408.4	233.3	–	188.7
$B(E2: 0_\tau^+ \to 2_1^+)$	0.0	–	0.0	0.0	<0.02	0.0	0.0	1.64	0.0	0.0	0.0	0.01	0.0	0.0	0.0
$B(E2: 2_3^+ \to 0_\zeta^+)$	72.2	60.1	64.3	122.0	119.2	79.5	–	256.3	105.0	104.5	73.4	209.0	108.9	–	76.1
$B(E2: 0_\zeta^+ \to 2_1^+)$	86.8	42.3	83.7	182.7	105.1	121.8	–	89.4	175.0	158.2	81.1	88.1	177.8	–	84.4

Notes on solutions to the standard pairing model

- Hamiltonian for pairing interaction in *non-degenerate* shells:

$$\hat{H} = \sum_j \varepsilon_j \hat{n}_j - G \sum_{jj'} \hat{S}_+^j \hat{S}_-^{j'}$$

- *Is the pairing model with non-degenerate orbits integrable?*

R.W. Richardson, Phys. Lett. **5** (1963) 82
M. Gaudin, J. Phys. (Paris) **37** (1976) 1087.

$$\left[\hat{S}_+^i, \hat{S}_-^j\right] = 2\delta_{ij}\hat{S}_z^j, \ \left[\hat{S}_z^i, \hat{S}_\pm^j\right] = \pm\delta_{ij}\hat{S}_+^j$$

Pairing with non-degenerate orbits

- A hamiltonian for pairing in *non-degenerate* shells is integrable! Solution:

$$\prod_{\alpha=1}^{k}\left(\sum_{j=1}^{n}\frac{1}{x_\alpha-2\varepsilon_j}S_+^j\right)|0\rangle,\ \ \rho_j=-(j+1/2)/2,$$

$$2G\sum_j\frac{\rho_j}{x_\alpha-2\varepsilon_j}+2G\sum_{\beta(\neq\alpha)}\frac{1}{x_\alpha-x_\beta}-1=0,\ \ \alpha=1,2,\dots,k.$$

- Let y(x) be a polynomial of degree k satisfying the following second order Fuchsian equation:

$$A(x)y''(x)+B(x)\,y'(x)-V(x)y(x)=0$$

where

$$A(x)=\prod_{i=1}^{n}(x-a_i)$$

$$B(x)/A(x)=\sum_{i=1}^{n}\frac{\rho_i}{(x-a_i)}$$

T J Stieltjes

V(x) is a polynomial of degree n-2 need to be determined

G. Szego, Amer. Math. Soc. Colloq. Publ. vol. 23, AMS,1975
T J Stieltjes, C. R. Acad Sci Paris 100 (1885) 439;620
E B Van Vleck, Bull. Amer. Math. Soc. 4 (1898) 426.

- Let y(x) be a polynomial of degree k with k zeros x_i:

$$y(x) = \prod_{i=1}^{k} (x - x_i)$$

One can easily check that y(x) satisfy

$$\frac{y''(x_i)}{y'(x_i)} = \sum_{j=1}^{k} \frac{2}{(x_i - x_j)}$$

$$A(x)y''(x)+B(x)\, y'(x)-V(x)y(x)=0$$

$$\Downarrow$$

$$\sum_{j=1}^{k} \frac{2}{(x_i - x_j)} + \sum_{\mu=1}^{n} \frac{\rho_\mu}{x_i - a_\mu} = 0, i = 1,2,\ldots,k.$$

- The corresponding Fuchsian equation is
  A(x)y″(x)+B(x) y′(x)-V(x)y(x)=0

$$A(x)=\prod_{j=1}^{n}(x-2\varepsilon_j),\, B(x)/A(x)=\sum_{j=1}^{n}\frac{2\rho_j}{x-2\varepsilon_j}-1/G$$

The number of solutions should be equal exactly to

$$\eta(n,k)=\sum_{p_1=0}^{-2\rho_1}\sum_{p_2=0}^{-2\rho_2}\cdots\sum_{p_n=0}^{-2\rho_n}\delta_{q,k},\quad q=\sum_{i=1}^{n}p_i.$$

When ρ_i=-1/2 corresponding to the Nilsson model, we have

$$\eta(n,k)=n!/((n-k)!k!).$$

The Heine-Stieltjes correspondence

Heine-Stieltjes and Van Vleck Polys

$$A(x)y''(x)+B(x)\,y'(x)-V(x)y(x)=0$$

$$A(x)=\prod_{j=1}^{n}(x-2\varepsilon_j),\quad B(x)/A(x)=\sum_{j=1}^{n}\frac{2\rho_j}{x-2\varepsilon_j}-1/G$$

$$y(x)=\sum_{j=0}^{k}a_j x^j,\quad V(x)=\sum_{j=0}^{n-1}b_j x^j$$

One will get two matrix equations:

$\mathbf{Fv}=b_0\mathbf{v}$, ⟵ $\mathbf{Pv}=0$, where $\mathbf{v}=\{a_0, a_1,\ldots, a_k\}$.

solves $\{a_0, a_1,\ldots, a_k\}$
And b_0.

provides unique solution for $\{b_1, b_2,\ldots, b_{n-1}\}$ as a function of $\{a_0, a_1,\ldots, a_k\}$.

is different from that shown in A. Faribault et al, PRB **83**, 235124 (2011) using Ricatti type equations.

Advantage of the procedure:

No divergence occurs in variations of both $\{a_0, a_1, \ldots, a_k\}$ and $\{b_0, b_1, \ldots, b_{n-1}\}$ in solving the two equations

$$\mathbf{F}\mathbf{v} = b_0 \mathbf{v} \quad \text{and} \quad \mathbf{P}\mathbf{v} = 0, \text{ where } \mathbf{v} = \{a_0, a_1, \ldots, a_k\},$$

$\mathbf{F}$: $(k+1)\times(k+1)$ $\qquad$ $\mathbf{P}$: $(n-1)\times(k+1)$

so any recursive and iteration method can be used.

X. Guan, K.D. Launey, M. Xie, L. Bao, F. Pan, and J. P. Draayer, **Phys. Rev. C 86 (2012) 024313; Comp. Phys. Commun. 185 (2014) 2714**.

However, numerical work in finding roots of the Gaudin-Richardson equations increases with the increasing of the number of orbits and the number of valence nucleon pairs, which limits the application of the theory

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Algebraic Solutions of an Extended Pairing Model for Well Deformed Nuclei

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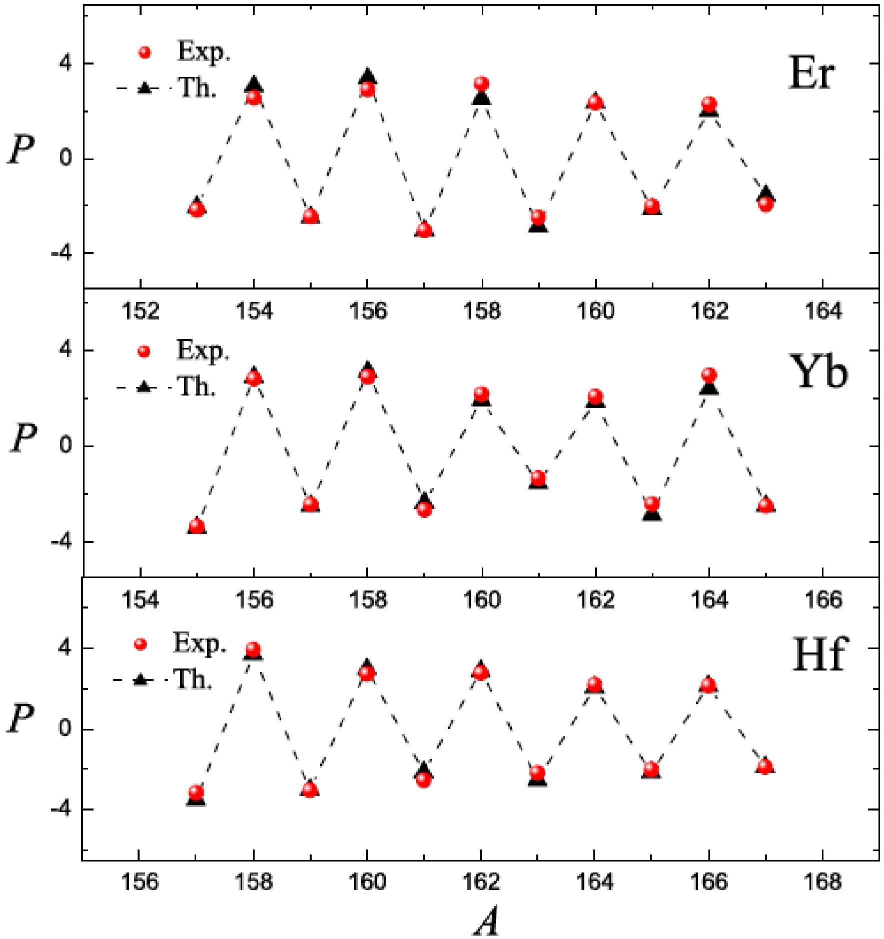
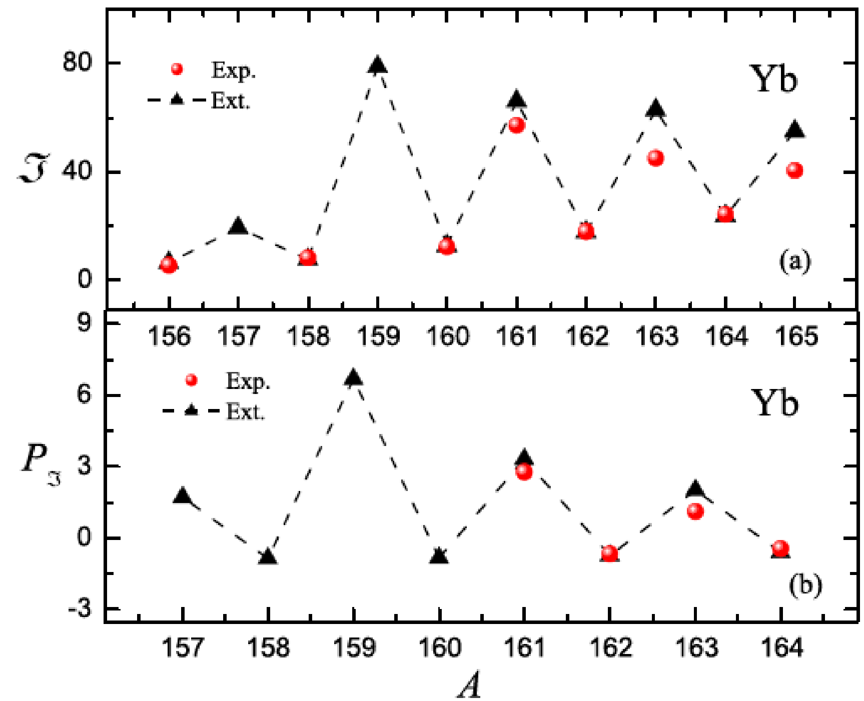
A Nilsson mean-field plus extended pairing interaction Hamiltonian with many-pair interaction terms is proposed. Eigenvalues of the extended pairing model are easy to obtain. Our investigation shows that one- and two-body interactions continue to dominate the dynamics for relatively small values of the pairing strength. As the strength of the pairing interaction grows, however, the three- and higher many-body interaction terms grow in importance. A numerical study of even-odd mass differences in the $^{154-171}$Yb isotopes shows that the extended pairing model is applicable to well deformed nuclei.

$$\hat{H} = \sum_{j=1}^{p} \epsilon_j n_j - G \sum_{i,j=1}^{p} a_i^+ a_j - G\left(\sum_{\mu=2}^{\infty} \frac{1}{(\mu!)^2} \sum_{i_1 \neq i_2 \neq \cdots \neq i_{2\mu}} a_{i_1}^+ a_{i_2}^+ \cdots a_{i_\mu}^+ a_{i_{\mu+1}} a_{i_{\mu+2}} \cdots a_{i_{2\mu}} \right)$$





$$P(A) = E_B(A+1) + E_B(A-1) - 2E_B(A)$$

$$\Im = 2\hbar^2 \sum_n \frac{|\langle n|J_{x'}|0\rangle|^2}{E_n - E_0}$$

$$P_{\Im} = \frac{\delta \Im}{\Im}\Big|_{\rm av} = \frac{\Im(A) - \frac{1}{2}[\Im(A+1) + \Im(A-1)]}{\frac{1}{2}[\Im(A+1) + \Im(A-1)]}$$

3. The extended monopole pairing

$$\hat{H} = \sum_j \epsilon_j N_j - \sum_j G_j S_j^\dagger S_j + \hat{H}_P$$

$$\hat{H}_P = -G \sum_{k=1}^{\infty} \sum_{j_1 \le \cdots \le j_k} \tilde{S}_{j_1}^\dagger \cdots \tilde{S}_{j_k}^\dagger \sum_{j_1' \le \cdots \le j_k'} \tilde{S}_{j_1'} \cdots \tilde{S}_{j_k'},$$

$$\tilde{S}_j^\dagger = \frac{1}{\sqrt{(Q_j - \hat{S}_j^0 + 1)(Q_j + \hat{S}_j^0)}} S_j^\dagger$$

$$Q_j = \tfrac{1}{2}(\Omega_j - \nu_j) \qquad \hat{S}_j^0 = \tfrac{1}{2}(N_j - \Omega_j)$$

$$C^{(\zeta)}_{n_1,\cdots,n_p} = \frac{1}{F(n_1,\cdots,n_p)},$$

$$-G\sum_{n_1=0}^{2Q_{j_1}}\cdots\sum_{n_p=0}^{2Q_{j_p}}\frac{\delta_{q,N}}{F(n_1,\cdots,n_p)} = 1.$$

$$|Q_j,-Q_j+n;\nu_j\rho_j\rangle = \tilde{S}_j^{\dagger n}|\nu_j\rho_j\rangle$$

$$|N,\zeta\rangle = \sum_{n_1,\cdots,n_p} C^{(\zeta)}_{n_1,\cdots,n_p}|n_1,\cdots,n_p\rangle = \sum_{n_1=0}^{2Q_{j_1}}\cdots\sum_{n_p=0}^{2Q_{j_p}}\delta_{\sum_{i=1}^p n_i,N}C^{(\zeta)}_{n_1,\cdots,n_p}|n_1,\cdots,n_p\rangle,$$

$\{|n_1,\cdots,n_p\rangle = \tilde{S}_{j_1}^{\dagger n_1}\cdots\tilde{S}_{j_p}^{\dagger n_p}|\nu_{j_1}\rho_{j_1},\cdots,\nu_{j_p}\rho_{j_p}\rangle\}$, **which are mutually orthonormal,**

$$\dim(V_N) = \sum_{n_1=0}^{2Q_{j_1}}\cdots\sum_{n_p=0}^{2Q_{j_p}}\delta_{\sum_{i=1}^p n_i,N}.$$

$$F(n_1,\cdots,n_p) = E^{(\zeta)} - G - \sum_{j=1}^{p}\epsilon_{j_i}(2n_{j_i}+\nu_{j_i}) + \sum_{i=1}^{p}G_{j_i}n_i(2Q_{j_i}-n_i+1),$$

TABLE I: The G values used in (18) with $G_j = G_s \ \forall \ j$ to produce the same ground state energy of the standard pairing for the seniority zero cases and the corresponding overlaps $|\langle Q^0, N|N, \zeta = 1\rangle|$ for given total number of pairs N.

N	1	2	3	4	5	6	7	8	9	10	11	12	
G/G_s	2.266	0.745	0.236	0.112	0.055	0.0304	0.0187	0.0126	0.0093	0.00743	0.00641	0.00596	
overlap	0.99	0.93	0.93	0.88	0.86	0.84	0.82	0.81	0.80	0.79	0.79	0.78	
N	13	14	15	16	17	18	19	20	21	22	23	24	25
G/G_s	0.00596	0.0064	0.00743	0.0093	0.0126	0.01868	0.0304	0.0550	0.112	0.262	0.712	2.266	1.000
overlap	0.78	0.79	0.79	0.80	0.81	0.82	0.84	0.86	0.88	0.90	0.94	0.99	1.00

with 11 j-orbits: $1s_{1/2}$, $1p_{3/2}$, $1p_{1/2}$, $1d_{5/2}$, $2s_{1/2}$, $1d_{3/2}$, $1f_{7/2}$, $2p_{3/2}$, $1f_{5/2}$, $2p_{1/2}$, and $1g_{9/2}$.

TABLE II: The overlap $|\langle Q^0, N|N, \zeta = 1\rangle|$ for given total number of pairs N with $G_j = G_s \ \forall \ j$ and $G = G_s$.

N	1	2	3	4	5	6	7	8	9	10	11	12	
Overlap	0.98	0.94	0.92	0.89	0.87	0.85	0.83	0.82	0.81	0.80	0.80	0.80	
N	13	14	15	16	17	18	19	20	21	22	23	24	25
Overlap	0.80	0.80	0.80	0.81	0.82	0.83	0.85	0.87	0.89	0.91	0.94	0.97	1.0

APPLICATION TO THE GROUND STATES OF $^{12-28}$O

with 11 j-orbits: $1s_{1/2}$, $1p_{3/2}$, $1p_{1/2}$, $1d_{5/2}$, $2s_{1/2}$, $1d_{3/2}$, $1f_{7/2}$, $2p_{3/2}$, $1f_{5/2}$, $2p_{1/2}$, **and** $1g_{9/2}$.

TABLE III: The neutron single-particle energies for ^{16}O generated from the spherical shell model.

Shell	Neutron energy (MeV)
$1s_{1/2}$	-30.316
$1p_{3/2}$	-15.834
$1p_{1/2}$	-9.910
$1d_{5/2}$	-0.860
$2s_{1/2}$	2.590
$1d_{3/2}$	7.781
$1f_{7/2}$	13.500
$2p_{3/2}$	20.160
$1f_{5/2}$	23.860
$1p_{1/2}$	24.608
$1g_{9/2}$	28.207

$E_{\rm B} = E_{\rm core} + E_k^{(1)}$, for the binding energy.

$$G_j = (18/A)^{1/3}(2.1246 + 2.8197/3 + 2.1845/2)/3 = G_s \ \forall \ j \ \textbf{(in MeV)}$$

which is the average value of the J=0 two-body matrix Elements of the universal ds-shell Hamiltonian of Wildenthal.

$$G_s = (18/12)^{1/3}(2.1246+2.8197/3+2.1845/2)/3 = 1.5861 \ \textbf{(in MeV)}$$

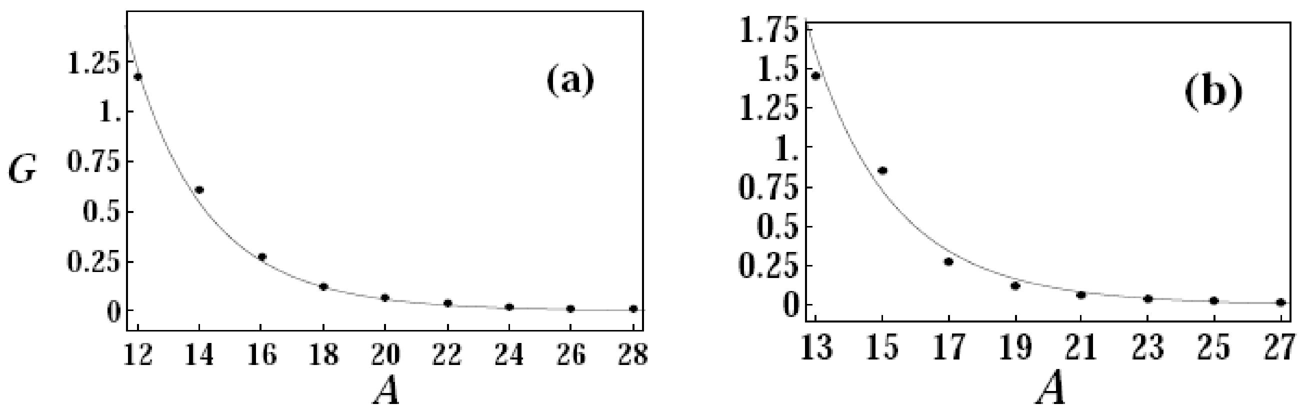
from which we get $G = 1.18\text{MeV}$ for ^{12}O.

From which we get $E_{\text{core}} = 47.375\text{MeV}.$

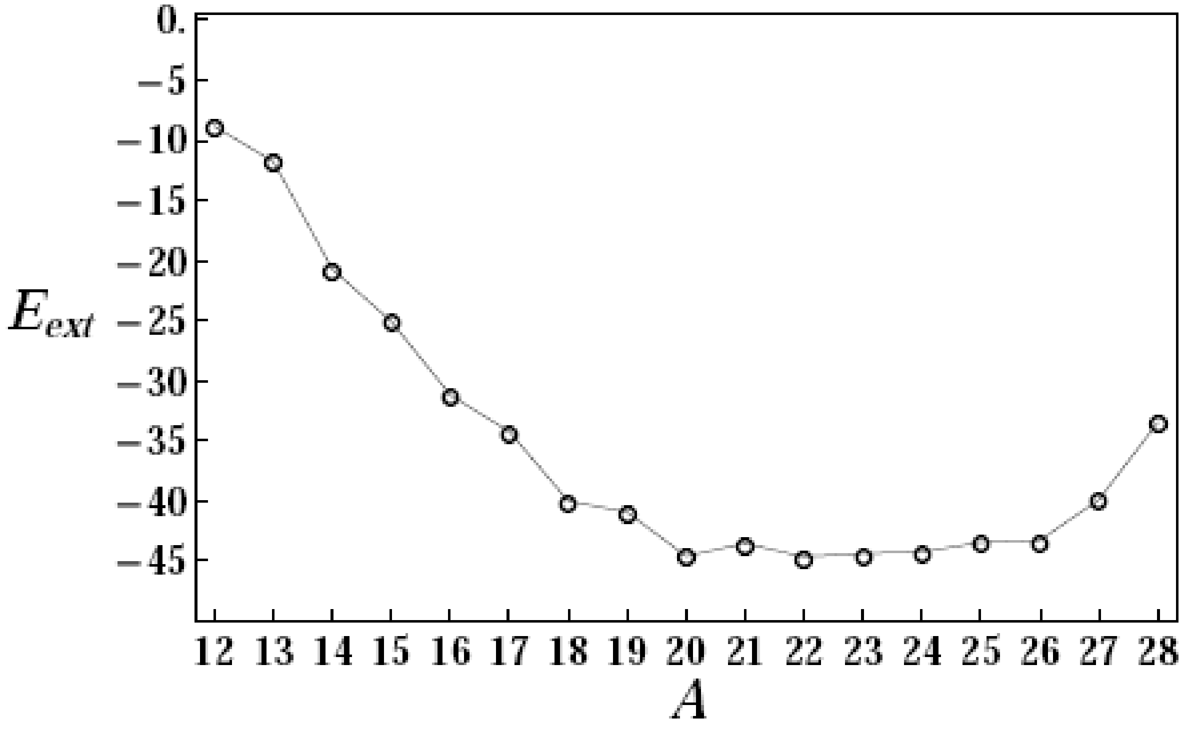
Since the binding energies are produced by adjusting the parameter G, the even-odd mass difference $P(A) = E_B(A) - E_B(A+1) - E_B(A-1)$ is simply given by the difference of the ground state energies obtained from (15) with $P(A) = E_A^{(1)} - E_{A+1}^{(1)} - E_{A-1}^{(1)}$ according to (19), which produces the experimental value of the even-odd mass differences for $^{12-28}$O exactly.

TABLE IV: The number of valence neutron pairs N, the binding energy (BE) in MeV, the pairing interaction strength G (in MeV) used in the fitting for the binding energies of $^{12-28}$O, and the corresponding dimension of the subspace V_N, where the experimental binding energies of $^{12-28}$O are taken from [24].

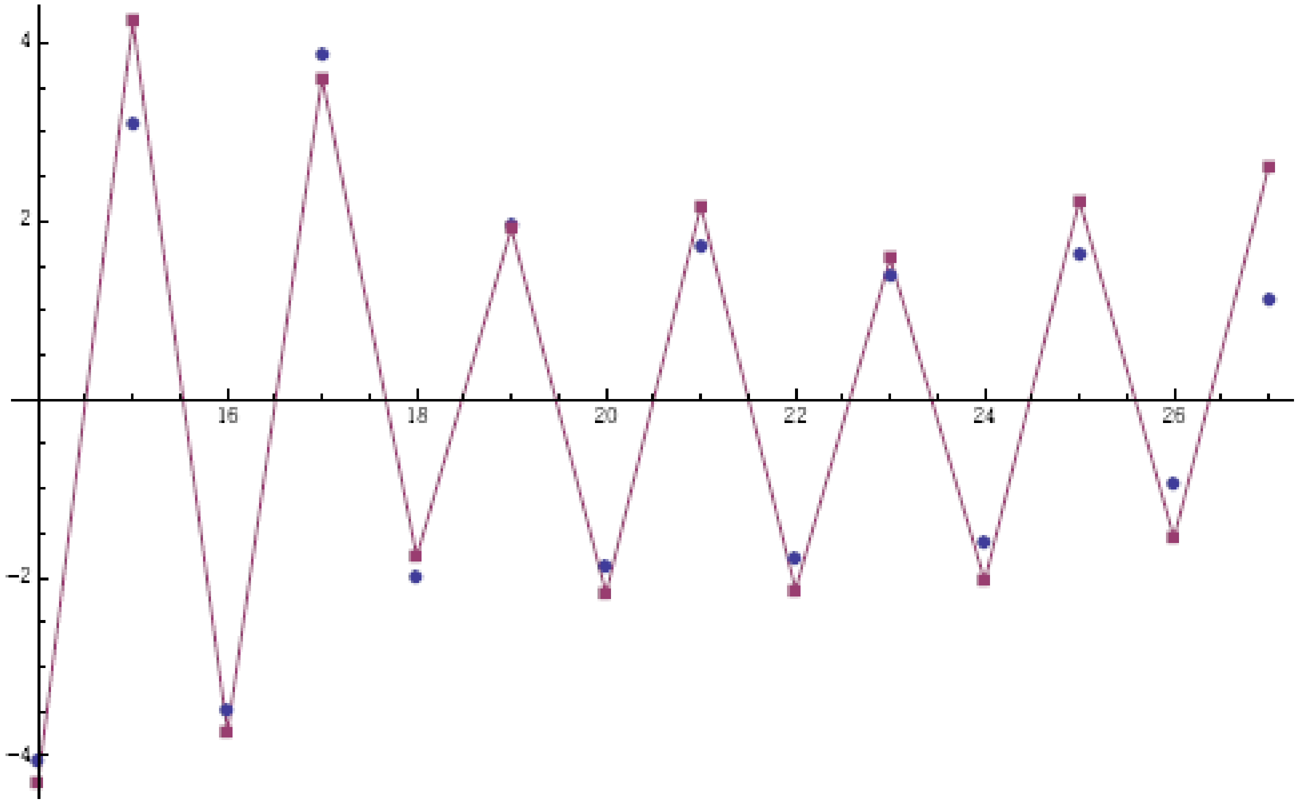
Nucleus	^{12}O	^{13}O	^{14}O	^{15}O	^{16}O	^{17}O	^{18}O	^{19}O	^{20}O
N	2	2	3	3	4	4	5	5	6
BE	58.68	75.553	98.728	111.945	127.616	131.75	139.806	143.754	151.36
G	1.18	1.46113	0.61307	0.86924	0.276678	0.28782	0.129469	0.1352595	0.0693543
dim(V_N)	62	61	239	187	708	698	1716	1665	3537
Nucleus	^{21}O	^{22}O	^{23}O	^{24}O	^{25}O	^{26}O	^{27}O	^{28}O	
N	6	7	7	8	8	9	9	10	
BE	155.169	162.008	164.772	168.96	168.175	168.87	167.211	167.664	
G	0.0734275	0.0416356	0.0456587	0.0273394	0.0340675	0.01921218	0.0246857	0.0146306	
dim(V_N)	3360	6361	5891	10160	8161	14594	11291	19019	



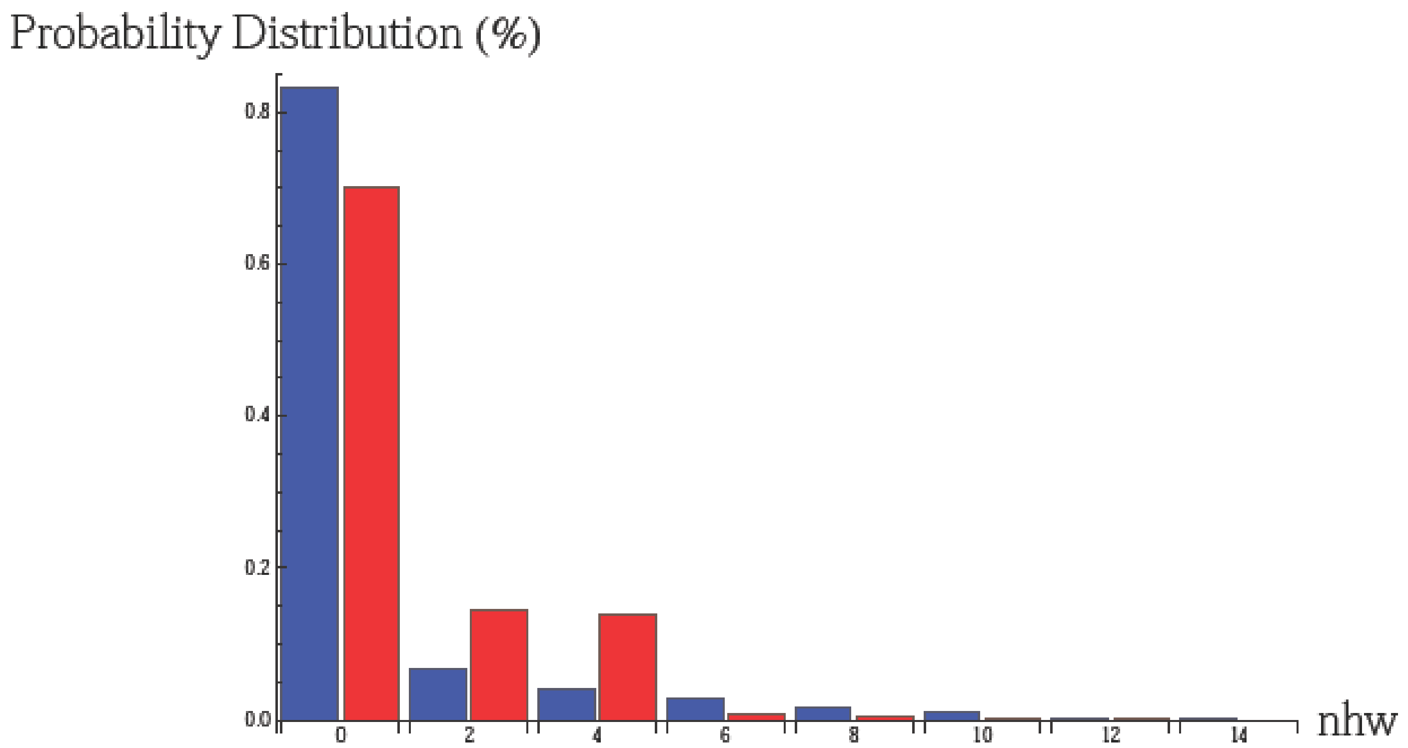
$$G = 3523.6(1 + 0.97\nu)\frac{e^{-A/4}}{A^2}\ \text{MeV}$$



The pairing energy contribution to the binding as a Function of the mass number A.



$$\frac{BE(A-2)-3BE(A-1)+3BE(A)-BE(A+1)}{4}$$

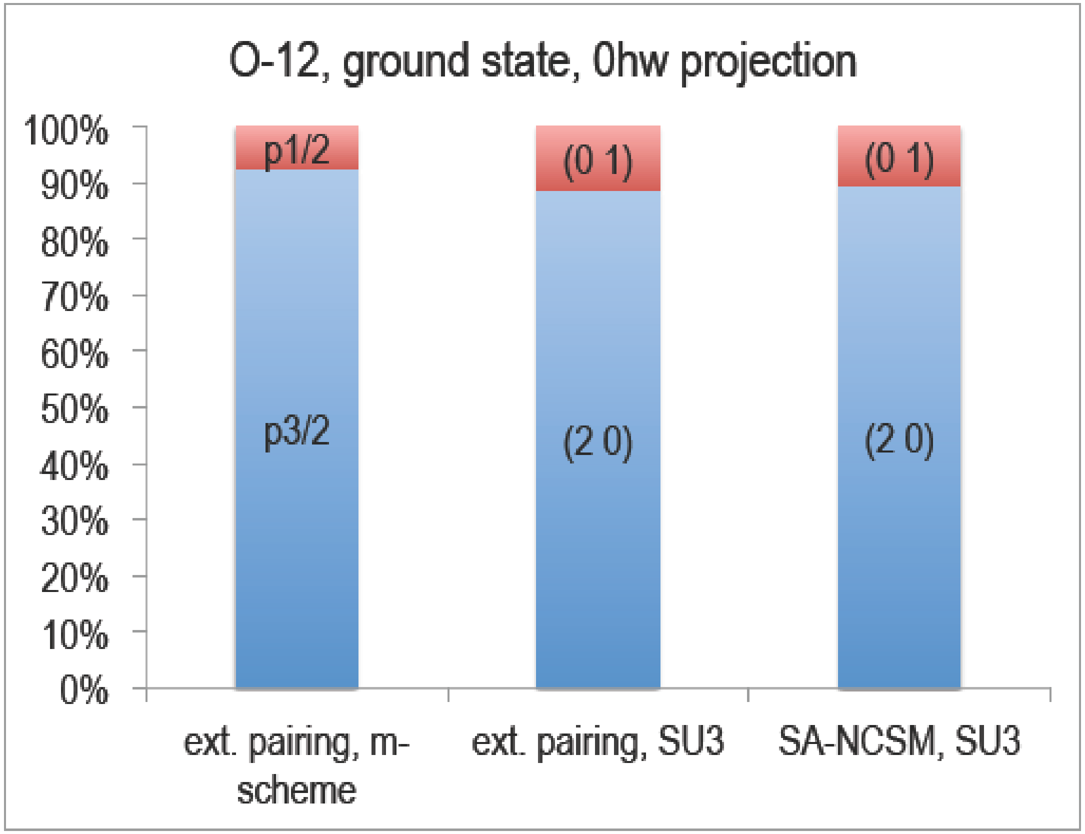


Probability distributions of the O-12 wavefunctions as a function of the number of the HO excitation quanta (shown along the horizontal axis, $n\hbar\omega = 0, 2, \ldots, 14\hbar\omega$)

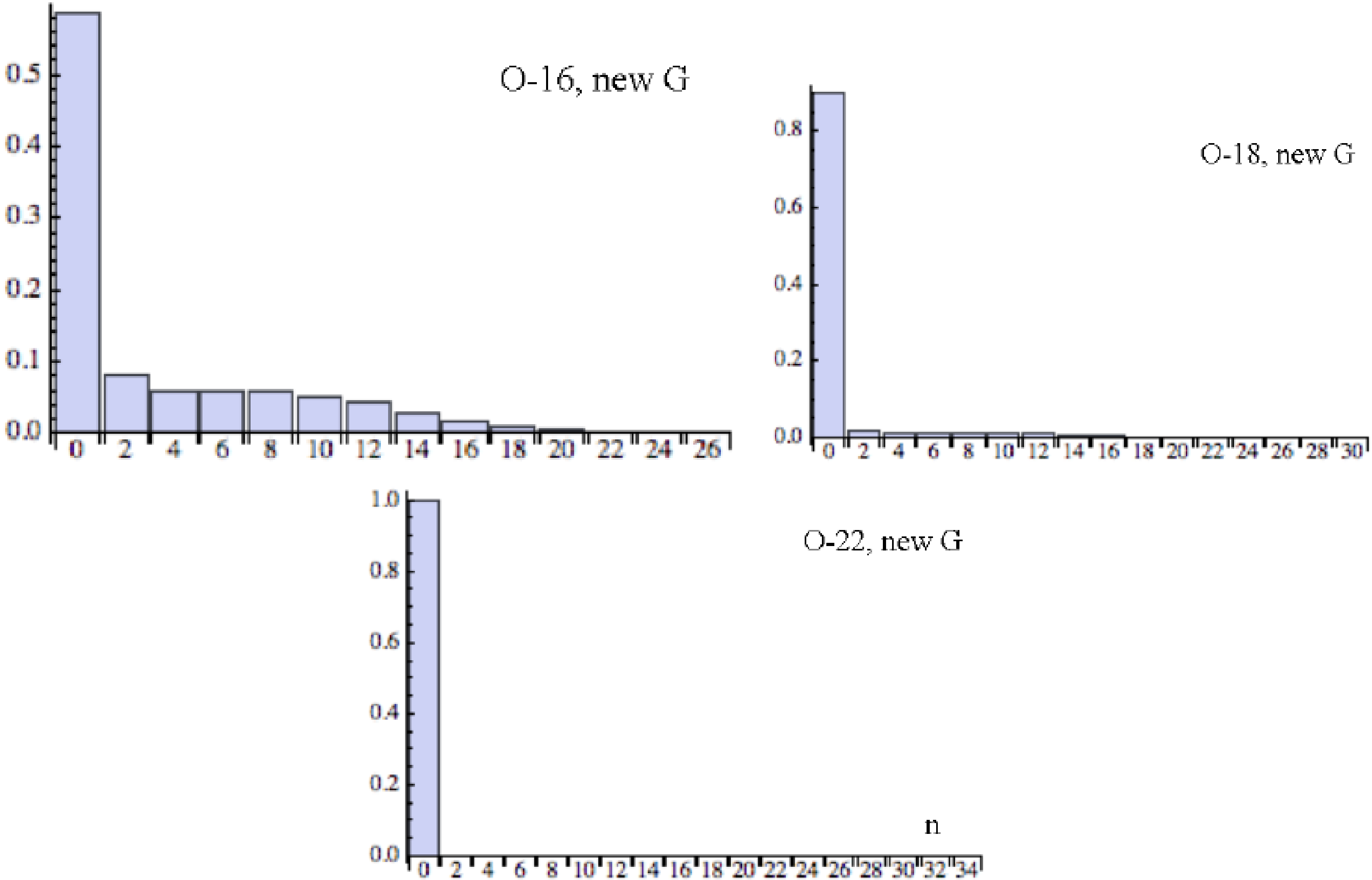
the present extended pairing model (blue)

the *ab initio* SA-NCSM (red)

with the bare JISP16 nucleon-nucleon interaction in the 14 HO shell model space and for $\hbar\omega = 15$ MeV



SU(3) content of the $0\hbar\omega$ part of the O-12 ground-state as compared to the corresponding *ab initio* 14-shell SA-NCSM results

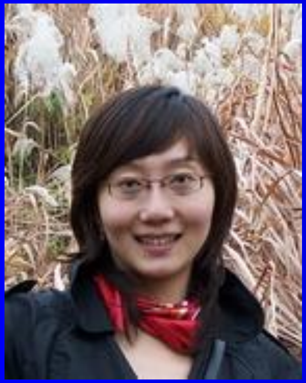

O-16, new G
0.5
0.4
0.3
0.2
0.1
0.0
0 2 4 6 8 10 12 14 16 18 20 22 24 26
O-18, new G
0.8
0.6
0.4
0.2
0.0
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30
O-22, new G
1.0
0.8
0.6
0.4
0.2
0.0
n
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34

Brief Summary

1) **We established an E2 algebra to construct exactly solvable models for both Boson and Fermion systems.**
2) **In application of the model for the IBM, an extended U(5)-O(6) Hamiltonian is constructed, which produces spectrum and E2 transition rates more close to those obtained from the collective E(5) model.**
3) **An exactly solvable spherical mean-field plus the extended monopole pairing up to infinite order is thus established, which includes the extended pairing model for deformed mean-field theory as a special case.**
4) **Our analysis shows that the model results with 11 j-orbits for $^{12-28}$O are consistent with those of the ab initio SA-NCSM.**

Contributors to this work also include


Kristina D. Launey, Jerry P. Draayer (LSU)



Xin Guan, Yu Zhang, Lianrong Dai (LNNU)

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Tank you !