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Multiple multi-orbit fermionic and bosonic pairing and rotational SU(3) algebras

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Plan of the Talk

- **Introduction**

- **Multiple pairing SU(2) and complimentary Sp(2Ω) algebras in SM :** selections rules for EM operators and applications

- **Multiple pairing SU(1,1) and complimentary SO(2Ω) algebras in IBM :** form of EM operators

- **Multiple SU(3) algebras in SM and IBM:** geometric shapes

- **Conclusions**

1. INTRODUCTION

Pairing algebras and the resulting seniority quantum number continue to play an important role in SM

For many single-*j* shell nuclei seniority quantum number is good or useful

for single-*j* shell: $H_p = -G\, S_+(j) S_-(j)$

$$S_+(j) = \sum_{m>0} (-1)^{j-m} a^\dagger_{jm} a^\dagger_{j-m} = \frac{\sqrt{2j+1}}{2} \left(a^\dagger_j a^\dagger_j \right)^0$$

$$S_-(j) = \left(S_+(j) \right)^\dagger,\; S_0(j) = \frac{\left(n_j - \Omega_j \right)}{2};\; \Omega_j = \frac{(2j+1)}{2}$$

$\Rightarrow \mathrm{SU_Q(2)} \rightarrow$ defines seniority quantum number

Pairing symmetry with nucleons occupying several *j*-orbits is less well understood from the point of view of its goodness and usefulness .

For multi- *j* shell nuclei, taking S_+ to be a sum of $S_+(j)$ with arbitrary phases – SU(2) algebra for each set of phases - multi-orbit or generalized seniority (AM choice).

With *r* number of *j*-orbits, 2^{r-1} SU(2) algebras

There are new selection rules for EM

All these extend to IBM's

Also, multiple SU(3) algebras both in SM and IBM's.

2. Multiple Pairing SU(2) and Complimentary Sp(2Ω) Algebras in SM :

Selections rules for EM operators and applications

Multiple multi-orbit pairing SU(2) algebras

$$S_+ = \sum_j \alpha_j S_-(j),\; S_- = \left(S_+\right)^\dagger,\; S_0 = \frac{(n-\Omega)}{2};\; \Omega=\sum_j \Omega_j$$

$\Rightarrow$ generalized quasi-spin $SU_Q(2)$ algebra

if $\alpha_j^2 = 1$ for all j

For each $\left\{\alpha_{j_1}, \alpha_{j_2}, ---, \alpha_{j_r}\right\}$ set there is a $SU_Q(2)$

Thus,with r number of orbits there are 2^{r-1} $SU_Q(2)$ algebras

$$\begin{aligned} \langle S_+S_-\rangle^{sm_s} &= = \langle S_+S_-\rangle^{mv} = \langle mv \mid S_+S_- \mid mv\rangle \\ &= \tfrac{1}{4}(m-v)(2\Omega - m - v + 2)\;, \end{aligned}$$

$$|m,v,\beta\rangle = \sqrt{\frac{(\Omega-v-p)!}{(\Omega-v)!p!}}\,(S_+)^{\frac{m-v}{2}}\,|v,v,\beta\rangle\;;\; p = \frac{(m-v)}{2}\;.$$

Complimentary *Sp(N)* algebra

In a given $\left(j_1, j_2, ---, j_r\right)^m$ space - $U(N)$ algebra:

$u_q^k(j_1, j_2) = \left(a_{j_1}^{\dagger} \tilde{a}_{j_2}\right)_q^k$ generators

irrep $\left\{1^m\right\}$

$$C_2(U(N)) = \sum_{j_1, j_2} (-1)^{j_1 - j_2} \sum_k u_q^k(j_1, j_2) \bullet u_q^k(j_2, j_1)$$

$$\left\langle C_2(U(N)) \right\rangle^m = m(N+1-m)$$

$$N = \sum_j (2j+1) = 2\Omega$$

more importantly, $U(N) \supset Sp(N)$

$Sp(N)$ generators: irrep $\langle 1^{\text{v}} \rangle$

$u_q^k(j, j)$ with k odd

$$V_q^k(j_1, j_2) = [R(j_1, j_2, k)]^{1/2} \left[\left(a_{j_1}^{\dagger} \tilde{a}_{j_2} \right)_q^k + X(j_1, j_2, k) \left(a_{j_2}^{\dagger} \tilde{a}_{j_1} \right)_q^k \right];$$

$j_1 > j_2$ and $X(j_1, j_2, k) = \pm 1$

$Sp(N) \Leftrightarrow SU_Q(2)$ for a given $\{\alpha_{j_1}, \alpha_{j_2}, ---, \alpha_{j_r}\}$ set, if

$R(j_1, j_2, k) = (-1)^{k+1} \alpha_{j_1} \alpha_{j_2}$ and $X(j_1, j_2, k) = (-1)^{j_1 + j_2 + k} \alpha_{j_1} \alpha_{j_2}$

$$C_2(Sp(N)) = 2 \sum_j \sum_{k=\text{odd}} u^k(j, j) \cdot u^k(j, j) + \sum_{j_1 > j_2; k} V^k(j_1, j_2) \cdot V^k(j_1, j_2)$$

$$C_2(U(N)) - C_2(Sp(N)) = 4 S_+ S_- - n$$

Selection rules for EMs transitions

phase

$$T_q^{xL} = \sum_j \varepsilon_{j,j}^{xL} \left(a_j^\dagger \tilde{a}_j\right)_q^L + \sum_{j_1 > j_2} \varepsilon_{j_1,j_2}^{xL} \left[\left(a_{j_1}^\dagger \tilde{a}_{j_2}\right)_q^L + \frac{\varepsilon_{j_2,j1}^{xL}}{\varepsilon_{j1,j_2}^{xL}} \left(a_{j_2}^\dagger \tilde{a}_{j_1}\right)_q^L\right]$$

$$\frac{\varepsilon_{j2,j1}^{xL}}{\varepsilon_{j1,j_2}^{xL}} = (-1)^{j_1+j_2+L} \alpha_{j_1} \alpha_{j_2} \rightarrow T_0^0 \text{ w.r.t. } SU_Q(2)$$

$$\frac{\varepsilon_{j2,j1}^{xL}}{\varepsilon_{j1,j_2}^{xL}} = (-1)^{j_1+j_2+L+1} \alpha_{j_1} \alpha_{j_2} \rightarrow T_0^1 \text{ w.r.t. } SU_Q(2)$$

note: $\frac{\varepsilon_{j2,j1}^{EL}}{\varepsilon_{j1,j_2}^{EL}} = -(-1)^{l_1+l_2+j_1+j_2+L}, \quad \frac{\varepsilon_{j2,j1}^{ML}}{\varepsilon_{j1,j_2}^{ML}} = (-1)^{l_1+l_2+j_1+j_2+L}$

L even or odd

with $\alpha_j = (-1)^l$, T^{EL} will be T_0^1 and T^{ML} will be T_0^0

$$T_0^0 : \mathrm{v} \rightarrow \mathrm{v}, \quad T_0^1 : \mathrm{v} \rightarrow \mathrm{v} \pm 2, \quad \text{no change in } m$$

Correlations between H and H_p

sp orbits	interaction	m	α_j	$\zeta(H,H_p)$
${}^0g_{7/2}$, ${}^1d_{5/2}$, ${}^1d_{3/2}$, ${}^2s_{1/2}$, ${}^0h_{11/2}$	jj55-SVD	$2-30$	$(+,+,+,+,-)$	0.33-0.11
			$(+,+,+,-,-)$	0.26-0.09
			$(+,+,-,+,-)$	0.17-0.06
			$(+,+,-,-,-)$	0.13-0.04
			$(+,-,+,+,-)$	0.11-0.04
${}^0f_{5/2}$, ${}^1p_{3/2}$, ${}^1p_{1/2}$, ${}^0g_{9/2}$	jun45	$2-20$	$(+,+,+,-)$	0.42-0.21
			$(+,+,-,-)$	0.27-0.13
			$(+,-,+,-)$	0.15-0.07
			$(+,-,-,-)$	0.12-0.06
${}^0f_{7/2}$, ${}^1p_{3/2}$, ${}^0f_{5/2}$, ${}^1p_{1/2}$	gxpf1	$2-18$	$(+,+,+,+)$	0.36-0.33
			$(+,+,+,-)$	0.22-0.20
			$(+,-,+,+)$	0.13-0.12
			$(+,-,+,-)$	0.13-0.11
			$(+,-,-,-)$	0.11-0.10

$$\|O\|_m = \sqrt{\left\langle \tilde{O}^\dagger \tilde{O} \right\rangle^m}$$

$$\zeta(O_1,O_2) = \frac{\left\langle \tilde{O}_1^\dagger \tilde{O}_2 \right\rangle^m}{\|O_1\|_m \, \|O_2\|_m}$$

French's SDM
(Kota and Haq book)

Most recent due to
K.D. Launey,
J.P. Draayer ---

Applications

$$\beta_j = (-1)^{\ell_j}$$

v

High-spin isomers in Sn isotopes

^{116}Sn to ^{130}Sn B(E2;2,10^{+}→2,8^{+})

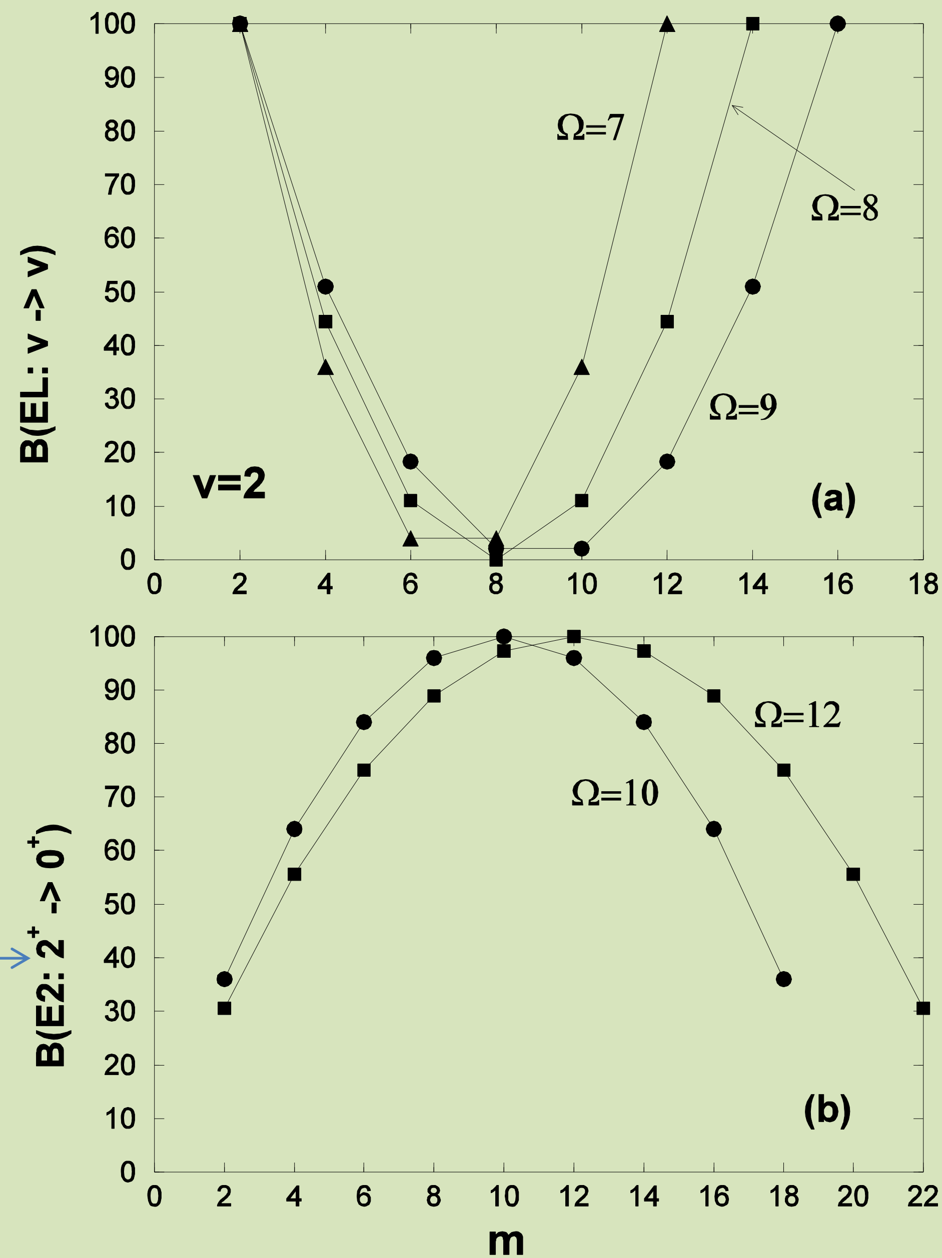
^{120}Sn to ^{128}Sn B(E2;4,15^{-}→4,13^{-})

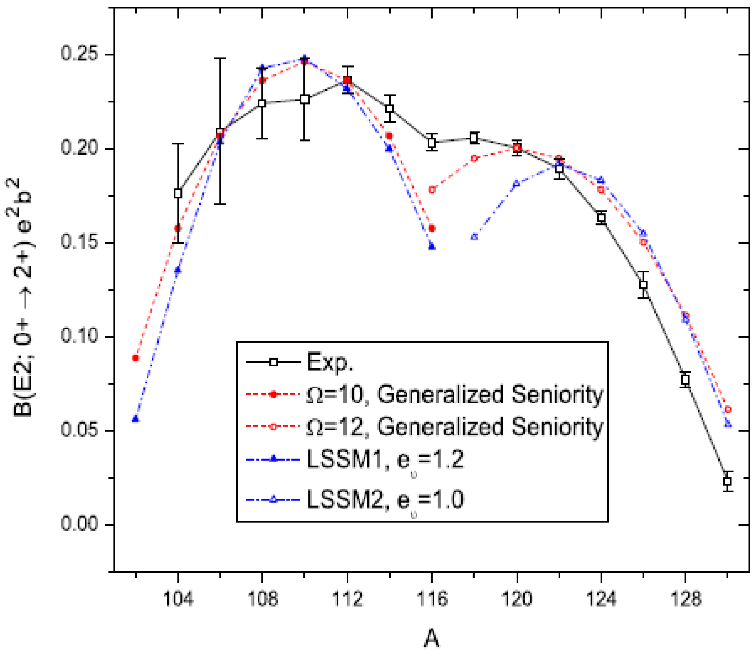
^{120}Sn to ^{126}Sn B(E1;4,13^{-}→4,12^{+})

$h_{11/2}, d_{3/2}, s_{1/2}$ with Ω=9

^{104}Sn to ^{116}Sn
$g_{7/2}, d_{5/2}, d_{3/2}, s_{1/2}$ with Ω=10 for Sn(104-116) and Sn(100) core

^{116}Sn to ^{130}Sn
$h_{11/2}, d_{5/2}, d_{3/2}, s_{1/2}$ with Ω=12 for Sn(116-130) and Sn(108) core





In summary, both the $B(E2; 2^+ \rightarrow 0^+)$ data and the $B(E2)$ and $B(E1)$ data for high-spin isomer states are explained by assuming goodness of generalized seniority with the choice $\beta_j = (-1)^{\ell_j}$ but with effective Ω values. Although the sp orbits (and hence Ω values) used are different for the low-lying levels and the high-spin isomer states, the good agreements between data and effective generalized seniority description on one hand and the correlation coefficients presented in Section IV on the other show that for Sn isotopes generalized seniority is possibly an 'emergent symmetry'.

3. Multiple pairing SU(1,1) and complimentary SO(2Ω) algebras in IBM:

form of EM operators

Multiple multi-orbit pairing SU(1,1) algebras

$$S_+^B = \sum_l \beta_l S_+^B(l),\; S_-^B = \left(S_+^B\right)^\dagger,\; S_0^B = \frac{\left(n^B + \Omega^B\right)}{2};$$

$$S_+^B(l) = \tfrac{1}{2} b_l^\dagger \cdot b_l^\dagger,\quad \Omega^B = \sum_l \Omega_l^B = \sum_l (2l+1)/2$$

$\Rightarrow$ generalized quasi-spin $SU_Q^B(1,1)$ algebra if $\beta_j^2 = 1$ for all l

For each $\left\{\beta_{l_1}, \beta_{l_2}, ---, \beta_{l_r}\right\}$ set a $SU_Q^B(1,1)$ – with r orbits 2^{r-1}

In $\left(l_1, l_2, ---, l_r\right)^m$ space - $U(N)$ algebra with $u_q^k(j_1, j_2) = \left(b_{j_1}^\dagger \tilde{b}_{j_2}\right)_q^k$

$$C_2(U(N)) = \sum_{l_1, l_2} (-1)^{l_1 + l_2} \sum_k u_q^k(l_1, l_2) \bullet u_q^k(l_2, l_1);\; N = \sum_l (2l+1) = 2\Omega$$

$$\left\langle C_2(U(N)) \right\rangle^{\{m\}} = m(N+1-m)$$

important: $U(N) \supset SO(N)$, $SO(N)$ is complimentary to $SU_Q^B(1,1)$

$SO(N)$ generators: irrep $\left[\omega^B\right]$

$u_q^k(l,l)$ with k odd, and with $l_1 > l_2$ and $Y(l_1,l_2,k) = \pm 1$

$$V_q^k(l_1,l_2) = [(-1)^{l_1+l_2}\,Y(l_1,l_2,k)]^{1/2}\left[\left(b_{l_1}^\dagger \tilde{b}_{l_2}\right)_q^k + Y(l_1,l_2,k)\left(b_{l_2}^\dagger \tilde{b}_{l_1}\right)_q^k\right];$$

$SO(N) \Leftrightarrow SU_Q^B(1,1)$ for a given $\left\{\beta_{l_1}, \beta_{l_2}, ---, \beta_{l_r}\right\}$ set, if

$$Y(l_1,l_2,k) = (-1)^{k+1}\,\beta_{l_1}\beta_{l_2}$$

$$C_2(SO(N)) = 2\sum_l \sum_{k=\text{odd}} u^k(l,l)\cdot u^k(l,l) + \sum_{j_1>j_2;k} V^k(l_1,l_2)\cdot V^k(l_1,l_2)$$

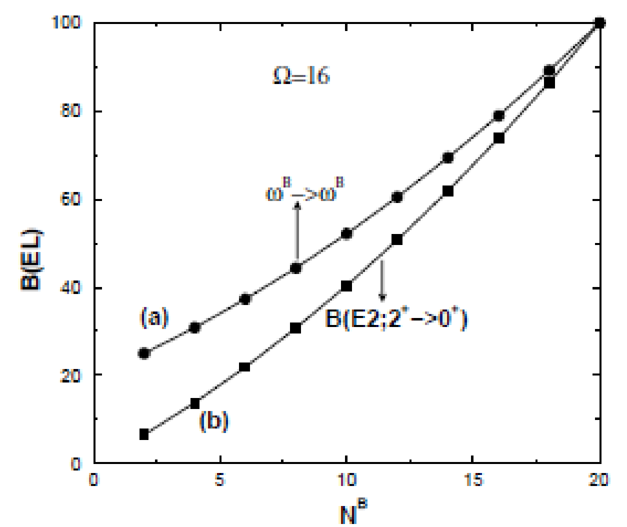
$$C_2(U(N)) - C_2(SO(N)) = 4S_+^B S_-^B - n^B$$

Selection rules for EM transitions

$$\frac{\epsilon^k_{\ell_2,\ell_1}}{\epsilon^k_{\ell_1,\ell_2}} = (-1)^{k+1}\beta_{\ell_1}\beta_{\ell_2} \rightarrow T_0^0\,, \quad \frac{\epsilon^k_{\ell_2,\ell_1}}{\epsilon^k_{\ell_1,\ell_2}} = (-1)^{k}\beta_{\ell_1}\beta_{\ell_2} \rightarrow T_0^1$$

Given β_l defining S_+ or H_p one can choose EM operators to be T_0^1 or T_0^0

often this is not followed in IBM's



Ru isotopes

$$(i)\ sdg\mathrm{IBM}:\ H_p = -GS_+S_-$$

$$S_+ = s^\dagger s^\dagger - d^\dagger \cdot d^\dagger - g^\dagger \cdot g^\dagger$$

$$SO(15) \Leftrightarrow SU^{+--}(1,1)$$

$$T^{E2} = \alpha_1(d^\dagger\tilde{d})^2_\mu + \alpha_2(g^\dagger\tilde{g})^2_\mu + \alpha_3(s^\dagger\tilde{d} + d^\dagger\tilde{s})^2_\mu + \alpha_4(d^\dagger\tilde{g} + g^\dagger\tilde{d})^2_\mu$$

$T_0^0 + T_0^1$ w.r.t. $SU^{+--}(1,1)$ and T_0^1 w.r.t. $SU^{+++}(1,1)$

$$(ii)\ spdf\mathrm{IBM}:\ S_+ = s^\dagger s^\dagger - d^\dagger \cdot d^\dagger \pm p^\dagger \cdot p^\dagger \pm f^\dagger \cdot f^\dagger \Rightarrow SU^{+-\pm\pm}(1,1)$$

$SU^{+,-,-,-}(1,1)$ Kusnezov

$$T^{E1} = \alpha_{sp}\underbrace{\left(s^\dagger\tilde{p} + p^\dagger\tilde{s}\right)^1_\mu}_{T_0^1} + \alpha_{pd}\underbrace{\left(p^\dagger\tilde{d} + d^\dagger\tilde{p}\right)^1_\mu}_{T_0^0} + \alpha_{df}\underbrace{\left(d^\dagger\tilde{f} + f^\dagger\tilde{d}\right)^1_\mu}_{T_0^0}$$

↑ **change sign will make the whole operator** T_0^0

interepolating different S_+ giving generalized Hp to study QPT and order-chaos transitions

also in analyzing spectroscopic data Pan Feng, Draayer, Jafarizadeh ---

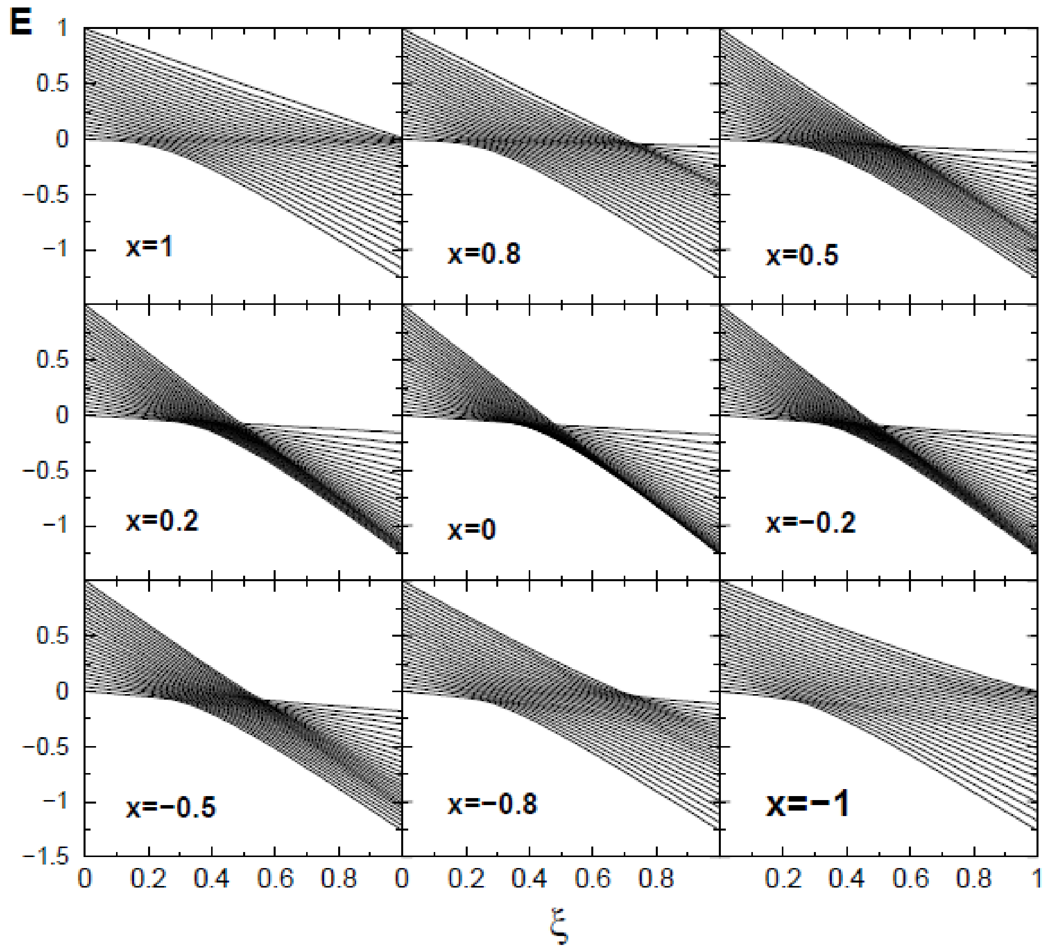


FIG. 3. Energy spectra for 50 bosons and $(\omega^B_{sd}, \omega^B_g) = (0, 0)$ in sdgIBM with the Hamiltonian $H_{sdg} = [(1-\xi)/N^B]\,\hat{n}_g + [(\xi/(N^B)^2]\,[4(S^{sd}_+ + xS^g_+)(S^{sd}_- + xS^g_-) - N^B(N^B+13)]$ where S^{sd}_+ is the S_+ operator for the sd boson system and S^g_+ for the g boson system. In each panel, energy spectra

4. Multiple SU(3) algebras in SM and IBM

Given an oscillator shell η, the *l* values are *l*=η, η-2, ---, 0 or 1

$$L_q^1 = \sum_\ell \sqrt{\frac{\ell(\ell+1)(2\ell+1)}{3}} \left(b_\ell^\dagger \tilde{b}_\ell\right)_q^1 ,$$

$$Q_q^2 = -(2\eta+3) \sum_\ell \sqrt{\frac{\ell(\ell+1)(2\ell+1)}{40(2\ell+3)(2\ell-1)}} \left(b_\ell^\dagger \tilde{b}_\ell\right)_q^2$$

$$+ \sum_{\ell<\eta} \alpha_{\ell,\ell+2} \sqrt{\frac{3(\ell+1)(\ell+2)(\eta-\ell)(\eta+\ell+3)}{20(2\ell+3)}} \left[\left(b_\ell^\dagger \tilde{b}_{\ell+2}\right)_q^2 + \left(b_{\ell+2}^\dagger \tilde{b}_\ell\right)_q^2\right] .$$

L_q^1, Q_q^2 form $SU(3)$ for $\alpha_{\ell,\ell+2} = \pm 1 \Rightarrow 2^{[\eta/2]}$ in number

Consider as an example sdgIBM – there will be 4 SU(3)'s –giving different shapes

$$|N;\beta_2;\beta_4,\gamma\rangle = \left[N!\left(1+\beta_2^2+\beta_4^2\right)^N\right]^{-1/2} \left\{s_0^\dagger + \beta_2\left[\cos\gamma\, d_0^\dagger + \sqrt{\tfrac{1}{2}}\sin\gamma\left(d_2^\dagger + d_{-2}^\dagger\right)\right] + \tfrac{1}{6}\beta_4\left[\left(5\cos^2\gamma+1\right)g_0^\dagger + \sqrt{\tfrac{15}{2}}\sin 2\gamma\left(g_2^\dagger + g_{-2}^\dagger\right) + \sqrt{\tfrac{35}{2}}\sin^2\gamma\left(g_4^\dagger + g_{-4}^\dagger\right)\right]\right\}^N |0\rangle .$$

$$Q^2_\mu(S) = \sqrt{\frac{8}{3}} Q^2_\mu = -11\sqrt{\frac{2}{21}}\,(d^\dagger\tilde{d})^2_\mu - 2\sqrt{\frac{33}{7}}\,(g^\dagger\tilde{g})^2_\mu$$

$$+\alpha_{sd}\, 4\sqrt{\frac{7}{15}}\, \left(s^\dagger\tilde{d} + d^\dagger\tilde{s}\right)^2_\mu + \alpha_{dg}\, \frac{36}{\sqrt{105}}\, \left(d^\dagger\tilde{g} + g^\dagger\tilde{d}\right)^2_\mu \;.$$

$$E_{SU_{sdg}(3)}\,(N;\beta_2,\beta_4,\gamma) = \langle N;\beta_2;\beta_4,\gamma \mid -Q^2(S)\cdot Q^2(S) \mid N;\beta_2;\beta_4,\gamma\rangle$$

$$= \frac{-N^2}{(1+\beta_2^2+\beta_4^2)^2}\,\left[\frac{448}{15}\,\alpha_{sd}^2\,\beta_2^2 + \frac{384\sqrt{14}}{35}\,\alpha_{sd}\alpha_{dg}\,\beta_2^2\,\beta_4\right.$$

$$+\frac{352\sqrt{35}}{105}\,\alpha_{sd}\,\beta_2^3\,cos3\gamma + \frac{64\sqrt{35}}{21}\,\alpha_{sd}\beta_2\,\beta_4^2\,cos3\gamma + \frac{3456}{245}\,\alpha_{dg}^2\,\beta_2^2\,\beta_4^2$$

$$+\frac{1056\sqrt{10}}{245}\,\alpha_{dg}\,\beta_2^3\,\beta_4\,cos3\gamma + \frac{484}{147}\,\beta_2^4 + \frac{192\sqrt{10}}{49}\,\alpha_{dg}\,\beta_2\,\beta_4^3\,cos3\gamma$$

$$\left. + \frac{880}{441}\,(4-cos^2 3\gamma)\,\beta_2^2\beta_4^2 + \frac{400}{1323}\,(16-7cos^2 3\gamma)\,\beta_4^4\right] \;.$$

Table 2. Equilibrium shapes for the four $SU(3)$ algebras in sdgIBM

α_{sd}	α_{dg}	β_2^0	β_4^0	γ^0	$E^0_{SU_{sdg}(3)}$
+1	+1	$\sqrt{20/7}$	$\sqrt{8/7}$	0°	$-(64/3)N^2$
-1	+1	$\sqrt{20/7}$	$-\sqrt{8/7}$	60°	$(-64/3)N^2$
+1	-1	$\sqrt{20/7}$	$-\sqrt{8/7}$	0°	$(-64/3)N^2$
-1	-1	$\sqrt{20/7}$	$\sqrt{8/7}$	60°	$(-64/3)N^2$

5. Conclusions

•In SM there will be multiple pairing SU(2) algebras in j-j coupling for identical nucleons with a complimentary Sp(N) algebra having number conserving operators

•They will give selection rules for EM operators

•Sn isotopes provide (with AM choice) examples – low-lying levels and high-spin isomers

•However, the correlation between realistic H and Hp is maximum 0.3

•There are multiple pairing SU(1,1) algebras in IBM's with corresponding SO(N) algebras

•SU(1,1) tensorial nature of EM operators can be determined consistently

•May give new insights into QPT, order-chaos transitions

•There are multiple SU(3) algebras in both SM and IBM – in IBM they will give different geometric shapes

•search for multiple pairing and SU(3) algebras is needed

•In future, further studies of multiple SU(3) algebras and also address multiple pairing algebras with internal quantum numbers (spin-isospin or isospin)