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Fock space dualities

K. Neergård

Næstved, Denmark

Abstract

The earliest example in physics of a Fock space duality appeared in 1928. Wigner and von Neumann showed that due to total antisymmetry, the wave function of a number of electrons in an atomic shell with azimutal quantum number l must factor into orbital and spin parts of conjugate symmetry. This may be seen as an $\mathfrak{gl}(2l+1)$–$\mathfrak{gl}(2)$ duality on the Fock space of the shell, where $\mathfrak{gl}(n)$ is the general linear Lie algebra over the complex numbers. In the wake of the nuclear BCS theory, Helmers proved in 1961 a theorem of $\mathfrak{sp}(d)$–$\mathfrak{sp}(2k)$ duality pertaining to a system of k kinds of fermion inhabiting a common single-kind state space of dimension d, and much later, in 2019, I proved an analogous $\mathfrak{o}(d)$–$\mathfrak{o}(2k)$ duality theorem. Here, $\mathfrak{sp}(n)$ and $\mathfrak{o}(n)$ are the symplectic and orthogonal subalgebras of $\mathfrak{gl}(n)$. Examples of $\mathfrak{sp}(2k)$ and $\mathfrak{o}(2k)$ include Kerman's quasispin algebra, which is isomorphic to $\mathfrak{sp}(2)$, the $\mathfrak{o}(5)$ Lie algebra of Flowers and Szpikowski, which is isomorphic to $\mathfrak{sp}(4)$, and the $\mathfrak{o}(8)$ Lie algebra of Flowers and Szpikowski, which has attracted recent attention related to questions of isospin $T = 0$ pairing in nuclei. These Lie algebras have in common that they connect nuclear states with different numbers of nucleons.

In mathematics, Howe proved in 1976 a very general duality theorem, which implies every duality mentioned above. Unlike the latter, Howe's theorem establishes 1–1 correspondence between the equivalence classes of irreducible representations of a group and a Lie superalgebra realised on the Fock space rather than between a couple of Lie algebras. The Lie superalgebra generally connects states with different numbers of particles. Special cases include, in an obvious notation, fermion and boson $\mathrm{GL}(d)$–$\mathfrak{gl}(k)$ dualities, fermion $\mathrm{Sp}(d)$–$\mathfrak{sp}(2k)$ and $\mathrm{O}(d)$–$\mathfrak{o}(2k)$ dualities and boson $\mathrm{Sp}(d)$–$\mathfrak{o}(2k)$ and $\mathrm{O}(d)$–$\mathfrak{sp}(2k)$ dualities. The complexification of the Lie algebra of the group "$\mathrm{Sp}(3,\mathbb{R})$" suggested by Rosensteel and Rowe to model nuclear collective motion is an example with $d = A-1$ and $k = 3$ of the Lie algebra in the $\mathrm{O}(d)$–$\mathfrak{sp}(2k)$ duality.

Among the special cases of Howe duality, the $\mathrm{GL}(d)$–$\mathfrak{gl}(k)$ dualities and the $\mathrm{Sp}(d)$–$\mathfrak{sp}(2k)$ duality follow almost trivially from the corresponding relations between Lie algebras. In 2020, I showed how to derive also the $\mathrm{O}(d)$–$\mathfrak{o}(2k)$ duality from the relation of Lie algebras, and soon thereafter, I obtained in this way also a hitherto undescribed $\mathfrak{o}(d)$–$\mathrm{Pin}(2k)$ duality, where $\mathrm{Pin}(n)$ is the double covering group of $\mathrm{O}(n)$ that provides spin representations. Together, the triple of $\mathfrak{o}(d)$–$\mathfrak{o}(2k)$, $\mathrm{O}(d)$–$\mathfrak{o}(2k)$ and $\mathfrak{o}(d)$–$\mathrm{Pin}(2k)$ dualities present a nice, almost symmetric picture.