Fock space dualities

K. Neergård
Næstved, Denmark

Abstract

The earliest example in physics of a Fock space duality appeared in 1928. Wigner and von Neumann showed that due to total antisymmetry, the wave function of a number of electrons in an atomic shell with azimutal quantum number \( l \) must factor into orbital and spin parts of conjugate symmetry. This may be seen as a \( gl(2l + 1) - gl(2) \) duality on the Fock space of the shell, where \( gl(n) \) is the general linear Lie algebra over the complex numbers. In the wake of the nuclear BCS theory, Helmers proved in 1961 a theorem of \( sp(d) - sp(2k) \) duality pertaining to a system of \( k \) kinds of fermion inhabiting a common single-kind state space of dimension \( d \), and much later, in 2019, I proved an analogous \( o(d) - o(2k) \) duality theorem. Here, \( sp(n) \) and \( o(n) \) are the symplectic and orthogonal subalgebras of \( gl(n) \). Examples of \( sp(2k) \) and \( o(2k) \) include Kerman’s quasispin algebra, which is isomorphic to \( sp(2) \), the \( o(5) \) Lie algebra of Flowers and Szpikowski, which is isomorphic to \( sp(4) \), and the \( o(8) \) Lie algebra of Flowers and Szpikowski, which has attracted recent attention related to questions of isospin \( T = 0 \) pairing in nuclei. These Lie algebras have in common that they connect nuclear states with different numbers of nucleons.

In mathematics, Howe proved in 1976 a very general duality theorem, which implies every duality mentioned above. Unlike the latter, Howe’s theorem establishes 1–1 correspondence between the equivalence classes of irreducible representations of a group and a Lie superalgebra realised on the Fock space rather than between a couple of Lie algebras. The Lie superalgebra generally connects states with different numbers of particles. Special cases include, in an obvious notation, fermion and boson \( GL(d) - gl(k) \) dualities, fermion \( Sp(d) - sp(2k) \) and \( O(d) - o(2k) \) dualities and boson \( Sp(d) - o(2k) \) and \( O(d) - sp(2k) \) dualities. The complexification of the Lie algebra of the group “\( Sp(3, \mathbb{R}) \)” suggested by Rosensteel and Rowe to model nuclear collective motion is an example with \( d = A - 1 \) and \( k = 3 \) of the Lie algebra in the \( O(d) - sp(2k) \) duality.

Among the special cases of Howe duality, the \( GL(d) - gl(k) \) dualities and the \( Sp(d) - sp(2k) \) duality follow almost trivially from the corresponding relations between Lie algebras. In 2020, I showed how to derive also the \( O(d) - o(2k) \) duality from the relation of Lie algebras, and soon thereafter, I obtained in this way also a hitherto undescribed \( o(d) - Pin(2k) \) duality, where \( Pin(n) \) is the double covering group of \( O(n) \) that provides spin representations. Together, the triple of \( o(d) - o(2k) \), \( O(d) - o(2k) \) and \( o(d) - Pin(2k) \) dualities present a nice, almost symmetric picture.