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The Generalized Richardson-Gaudin Models

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Abstract.

The Pairing Model hamiltonian was solved exactly by Richardson in the sixties. We present here the conditions of integrability for a wider variety of pairing hamiltonians for fermion and boson systems, together with the exact solution for their complete sets of eigenstates. A study of the exact solution for boson hamiltonians with repulsive pairing shows an unexpected quantum phase transition, suggesting a new mechanism for sd dominance in boson models of nuclei.

1 Introduction

Exactly solvable models (ESM) have played an important role in the understanding of several aspects of strongly correlated quantum many-body problems, some not accessible from standard many-body methods. The main property of the ESM is that they provide in analytic form the complete set of eigenstates, offering a unique tool with which to understand the physics involved.

A common feature of ESM in nuclear structure is that the model hamiltonian is written as a linear combination of the Casimir operators of a group decomposition chain, ideally representing the properties of a nuclear phase. Examples include the three dynamical symmetries of the Interacting Boson Model (IBM) [1], the Elliott $SU(3)$ model and the $SU(2)$ one-level Pairing Model (PM) [2].

Superconductivity is a phenomenon common to both nuclear and condensed matter systems. It was first recognized in the latter field that superconductivity can be described by a microscopic pairing hamiltonian in the BCS approximation. While the pairing hamiltonian has been extensively used in both fields of physics in the BCS or more sophisticated approximations, the exact solution of the model given by Richardson in the sixties [3] passed almost unnoticed until very recently. The Richardson solution was rediscovered in the process of improving the description of the disappearance of superconducting correlations in ultrasmall aluminium grains, which required a more accurate treatment than could be provided by Number Projected BCS [4]. Since then, much work has been done to generalize the model [5], while preserving its exact solvability, and to apply it to a variety of important physical systems. Applications have been reported to Bose condensates [6], interacting boson models [7], electrons in 2D lattices [5] and nuclear superconductivity [8].

In this contribution, we review the recent generalization of the PM to three families of exactly solvable models. We then consider a model of repulsive pairing between bosons, which is part of the so-called rational family, and show that the model displays a quantum phase transition to a state with macroscopic occupation of the lowest two boson levels only, implying a fragmentation of the condensate. These results suggest a new mechanism for sd dominance in the IBM, which is triggered by the repulsion between bosons that arises from the Pauli Principle at the underlying nucleon level.

2 Three Families of Exactly Solvable Pairing Models

We begin our discussion of the three families of exactly-solvable Pairing Models by defining the elementary operators of the pair algebra,

$$K_l^0 = \frac{1}{2}\sum_m a_{lm}^\dagger a_{lm} \pm \frac{1}{4}\Omega_l \tag{1}$$

$$K_l^\dagger = \frac{1}{2}\sum_m a_{lm}^\dagger a_{l\overline{m}}^\dagger = (K_l)^\dagger \tag{2}$$

which in turn are the generators of the $SU(2)$ group for fermions, or the $SU(1,1)$ group for bosons, and close the corresponding commutator algebras

$$\left[K_l^0, K_{l'}^+\right] = \delta_{ll'} K_l^+ \ , \quad \left[K_l^+, K_{l'}^-\right] = \mp 2\delta_{ll'} K_l^0 \tag{3}$$

The operator $K_l^\dagger$ in (2) creates a pair of particles in time reversed states with $a^\dagger(a)$ the particle creation (annihilation) operator and Ω_l the degeneracy of level l. Throughout the paper, the upper sign refers to bosons and the lower sign to fermions.

Alternatively, we may discuss the pair algebra in terms of the pair creation operator $A_l^\dagger = 2K_l^+$, the pair annihilation operator $A_l = 2K_l$ and the number operator $n_l = \mp\frac{1}{2}\Omega_l + 2K_l^0$.

Assuming that there are L single-particle levels and taking into account that each $SU(2)$ or $SU(1,1)$ group has one degree of freedom, a model built up from these operators is integrable if there are L independent hermitian and global operators that commute with one another. These operators are the quantum invariants and their eigenvalues, the constants of motion of the system, completely classify their common eigenstates. To find these operators, we first define the most general set of hermitian and number-conserving one- and two-body operators in terms of the K generators:

$$R_l = K_l^0 + 2g \sum_{l'(\neq l)} \frac{X_{ll'}}{2}\left(K_l^+ K_{l'}^- + K_l^- K_{l'}^+\right) \mp Y_{ll'} K_l^0 K_{l'}^0 \tag{4}$$

Up to this point, the matrices X and Y from which the R operators are defined are completely free. Here we fix them by imposing the condition that they must mutually commute to define an integrable model. The condition $[R_l, R_{l'}] = 0$ is fulfilled if the X and Y matrices are antisymmetric and satisfy

$$Y_{ij}X_{jk} + Y_{ki}X_{jk} + X_{ki}X_{ij} = 0 \tag{5}$$

An analogous condition was encountered by Gaudin [9] in a spin model now known as the Gaudin magnet. His model is based on R operators similar to (4), but without the one-body term. Solving (5) leads to three families of solutions [10]:

I. The rational model

$$X_{ll'} = Y_{ll'} = \frac{1}{\eta_l - \eta_{l'}} \tag{6}$$

II. The trigonometric model

$$X_{ll'} = \frac{1}{\sin(\eta_l - \eta_{l'})}\ , \quad Y_{ll'} = \cot(\eta_l - \eta_{l'}) \tag{7}$$

III. The hyperbolic model

$$X_{ll'} = \frac{1}{\sinh(\eta_l - \eta_{l'})}\ , \quad Y_{ll'} = \coth(\eta_l - \eta_{l'}) \tag{8}$$

In all three families, the η_l are an arbitrary set of non-equal real numbers. Any choice of these parameters within any of the three families leads to an integrable model and any combination of the corresponding R operators produces an integrable hamiltonian.

The rational model was proposed in ref. [11] to demonstrate the integrability of the PM hamiltonian. Indeed the PM hamiltonian can be obtained as a linear combination of its R operators, viz: $H_{PM} = 2\sum_l \varepsilon_l R_l^I(g,\varepsilon_l)$, plus an appropriate constant to give

$$H_P = \sum_l \varepsilon_l \widehat{n}_l + 2g \sum_{ll'} K_l^+ K_{l'} \tag{9}$$

For all three models, the exact eigenstates in the seniority zero subspace can be expressed as

$$|\Psi\rangle = \prod_{\alpha=1}^{M} B_\alpha^\dagger |0\rangle , \quad B_\alpha^\dagger = \sum_l u_l(e_\alpha)\ K_l^+ \tag{10}$$

where M is the number of pairs. The function u, which depends on a set of unknown *pair energies* e_α, must fulfill the L eigenvalue equations $R_i |\Psi\rangle = r_i |\Psi\rangle$. A similar ansatz can be used for the states with other seniorities.

Here we summarize the results for the rational model, which is used in the application of the next section. Details on the other two families can be found in ref. [5].

$$u_i(e_\alpha) = \frac{1}{2\eta_i - e_\alpha} \tag{11}$$

$$1 + g\sum_j \frac{\Omega_j}{2\eta_j - e_\alpha} \mp 4g \sum_{\beta(\neq\alpha)} \frac{1}{e_\alpha - e_\beta} = 0 \tag{12}$$

$$r_i = \pm\frac{\Omega_i}{4}\left[1 - \frac{g}{2}\sum_{j(\neq i)} \frac{\Omega_j}{\eta_i - \eta_j} \mp 4g\sum_\alpha \frac{1}{2\eta_i - e_\alpha}\right] \tag{13}$$

For a given set of the $L+1$ free parameters, η_i and g, of the model, one has to solve the coupled set of M nonlinear equations (12) for the pair energies e_α. There are as many independent solutions as states in the Hilbert space. For the particular case of the PM hamiltonian (9), the eigenstates are obtained by solving the set of nonlinear coupled equations (12) with the parameters η_l replaced by the single-particle energies ε_l. The eigenvalues of the PM hamiltonian are then given by the sum $E_{PM} = 2\sum_l \varepsilon_l r_l^I(g,\varepsilon_l)$, where $r_l(g,\varepsilon_l)$ are the eigenvalues of the R_l operators for the set of parameters g, ε_l (13). Working out the sum and using several relations that can be derived from (12), one finally arrives to the simple form for the PM eigenvalues

$$E_{PM} = \sum_\alpha e_\alpha \tag{14}$$

found long ago by Richardson [3] for the specific case of the PM hamiltonian.

3 First Applications to Boson Systems

We will first show that the PM hamiltonian with constant matrix elements (9) is quite unrealistic when describing confined boson systems. Looking back at the commutators of the pair operators K_l^+ in (3), we see that they are normalized to the square root of the degeneracy Ω_l of the level l which appears inside the definition of the K^0 operator in (1) . Thus, the hamiltonian (9) has an effective pairing interaction proportional to $\sqrt{\Omega_l \Omega_{l'}}$. In a spherical harmonic confining potential of dimension d, these degeneracies are in turn proportional to l^{d-1}, where l plays the role of the principal quantum number. On the other hand, the single-boson energies ε_l are linear in l. The net effect would then be to scatter boson pairs to high-lying levels with greater probability than to low-lying levels, producing unphysical occupation numbers. In order to test numerically this assertion, we have solved eq. (12) for a system of 1000 bosons ($M = 500$) trapped in a three-dimensional harmonic oscillator ($\Omega_l = (l+1)(l+2)/2$ and $\varepsilon_l = \hbar\omega(l+3/2)$)with a cutoff at $101/2\hbar\omega$ ($L = 50$ single-boson levels). Following Richardson [3], the occupation numbers can be calculated as

$$\langle n_l \rangle = \left\langle \frac{\partial H_P}{\partial \varepsilon_l} \right\rangle = \sum_\alpha \frac{\partial e_\alpha}{\partial \varepsilon_l} \tag{15}$$

From (15), and making use of(12) and (14), a set of M coupled linear equations in terms of M new unknowns can be obtained, which give the L occupation numbers. Details of the derivation can be found in ref. [3].

It is worthwhile to note here that the one-body density matrix is diagonal in the Richardson wave functions (6). A non-diagonal one-body term will either break a singlet boson pair or change the configuration of an unpaired boson making the expectation value vanish. In other words, non-diagonal one-body operators break the seniority quantum number.

In Figure 1, we show the occupation numbers versus the single-boson energies in units of $\hbar\omega$ for the pairing strength $g = -0.0025$. We have excluded the occupation of the $l = 0$ condensed boson because it lies outside the scale of the figure. The overall depletion is 0.21, which gives an occupation of the $l = 0$ state of $n_0 = 790$. The figure clearly shows the unphysical occupation of the high-lying levels due the effective pairing interaction discussed above.

We will use the freedom we have in choosing the parameters η entering in the definition of the R operators to obtain a more physical ESM. In order to cancel the dependence of the effective pairing interaction on the degeneracies, we make the following definition: $\eta_l = (\varepsilon_l)^3$. Then, the new hamiltonian becomes

$$H = \sum_l \varepsilon_l n_l + g \sum_{l \neq l'} \frac{1}{\varepsilon_l^2 + \varepsilon_l \varepsilon_{l'} + \varepsilon_{l'}^2} K_l \cdot K_{l'} \tag{16}$$

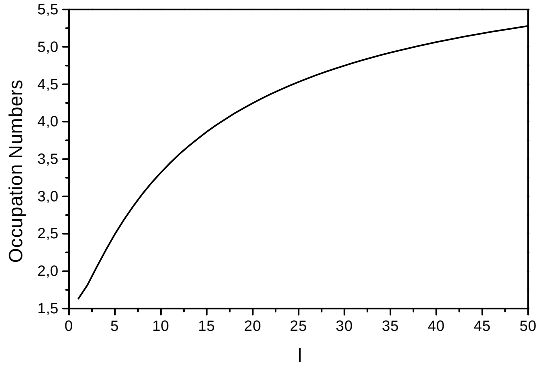

Figure 1. Occupation numbers for 1000 bosons in 50 harmonic oscillator shells for the PM hamiltonian with interaction strength $g = -0.0025$. The total depletion is 0.21. The occupation of the l=0 state is off scale.

Taking into account that ε_l is proportional to l, the new two-body term in (16)) cancels the dependence on the degeneracies of the effective pairing matrix elements. In expanding the sum of the R operators, we first replace the K^0 operator by the number operator to get the desired single-boson term, then add to the sum the term with $l = l'$, and then subtract it. The scalar product for $l = l'$ gives rise to the Casimir operator C_l^2 of $SU(1,1)$ with eigenvalue $C_l^2 = -\frac{\Omega_l}{4}\left(\frac{\Omega_l}{4} - 1\right)$. For the two-body interaction terms, we define the matrix $V_{ll'} = g/\left[2\left(\varepsilon_l^2 + \varepsilon_l\varepsilon_{l'} + \varepsilon_{l'}^2\right)\right]$. The final form of the hamiltonian is

$$H = \mathcal{E} + \sum_l \overline{\varepsilon}_l\, n_l + \sum_{ll\prime} V_{ll'}\left(A_l^\dagger A_{l'} - n_l n_{l'}\right) \tag{17}$$

with $\mathcal{E}$ an uninteresting constant and $\overline{\varepsilon}_l = \varepsilon_l + 2V_{ll} - \sum_{l'} V_{ll'}\,\Omega_{l'}$ being the single-boson energies.

It can be readily confirmed that now the effective interaction terms in (17)), contrary to (9), scale correctly with energy. The hamiltonian (17) has two-body matrix elements that decrease with the number of shells. It has the particular feature that the interactions of the pair- and density-fluctuations are the same in magnitude but opposite in sign. The latter feature may only be realized in particular situations whereas the equality of the magnitude of the interactions in both channels probably does not have any particular influence on the results presented below. The energy eigenvalues of (17) can be obtained by summing the eigenvalues (13) and are given by

$$E = \frac{1}{2}\sum_l \varepsilon_l\Omega_l - \frac{1}{4}\sum_{l\neq l'} V_{ll'}\Omega_l\Omega_{l'} - 2g\sum_{l\alpha}\frac{\varepsilon_l\Omega_l}{2\varepsilon_l^3 - e_\alpha}, \tag{18}$$

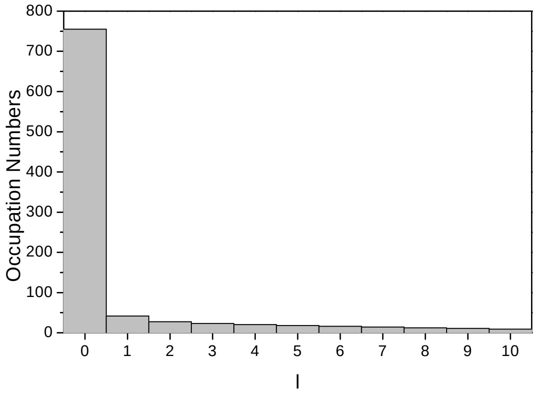

Figure 2. Occupation numbers for 1000 bosons in 50 harmonic oscillator shells and the hamiltonian (17). The interaction strength is $g = -1.0$, giving a depletion factor of 0.245.

We have solved eq. (12) with $\eta_l = (\varepsilon_l)^3$ for the same system as above $(M = 500,\ L = 50)$. For attractive pairing, all the magnitudes evolve smoothly with increasing attraction. In Figure 2, we show the occupation numbers for $g = -1.0$, corresponding to a depletion factor of 0.245. They display a reasonable physical pattern, filling first the lowest levels.

For repulsive pairing, we find an unexpected feature. In Figure 3, we show the occupation numbers of the first and second levels versus the scaled pairing interaction $x = \frac{2Mg}{\hbar\omega}$ for the same system ($M = 500,\ L = 50$). While the solid line corresponds to the exact solution, the dashed line represents the results of a Hartree-Fock (HF) treatment of the hamiltonian (17). For the critical value of the scaled interaction $x_c = 1$, the normal ground state boson condensate suddenly changes into a new phase in which the bosons occupy the $l = 0$ and the $l = 1$ levels while the occupation of the other levels is negligible. Differences between the exact solution and the HF approximation can only be seen in the two insets around the critical region, and we expect that HF is exact in the thermodynamic limit.

As mentioned above, the occupation numbers are the eigenvalues of the one-body density matrix. Therefore this new phase is a truly fragmented state, since the one-body density matrix has two macroscopic eigenvalues. Moreover the exact solution is rotationally invariant in both phases, while HF breaks rotational symmetry in the fragmented phase in order to describe the system by a single boson condensate. As a consequence, HF cannot fully describe the true features of this novel quantum phase transition.

It is commonly accepted since the work of Nozieres and Saint James [12] that for homogeneous systems fragmentation cannot occur in systems of scalar bosons with repulsive interactions. This might be the first example of fragmen-

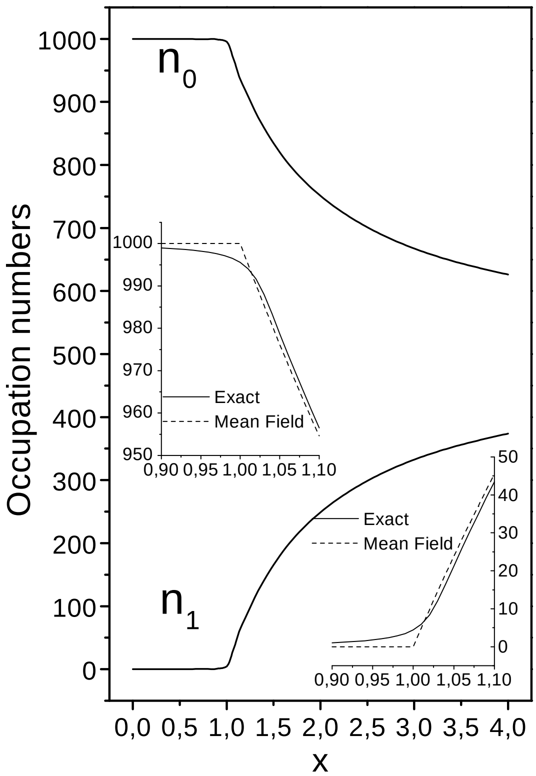

Figure 3. Occupation numbers n_0 and n_1 for 1000 bosons in 50 harmonic oscillator shells and hamiltonian (17) as a function of the adimensional parameter x.

tation in a confined boson system.

The complementary relation between fragmentation and spontaneous symmetry breaking unveiled in this application to a confined boson system may have important consequences to our understanding of Bose condensation.

4 A New Mechanism for sd Dominance in the IBM

Inherent in the success of the IBM [1] in describing the low energy properties of medium to heavy nuclei is the assumption of an effective decoupling of a subspace of collective bosons with angular momentum $L = 0\,(s)$ and $L = 2\,(d)$ from all other non-collective and higher-spin bosons. Despite much effort through the years to derive the IBM from the underlying nuclear shell model, there has only been success in the spherical vibrational regime. It is known from Hartree-Fock-Bogoliubov (HFB) studies that deformed nuclei are also dominated by SD fermion pairs, though with no clear decoupling from the rest of the space. It is worthwhile noting here that HFB only takes into account fermion pair correlations. We will show in this section that there is a mechanism at the two-body boson interaction level (four-fermion correlations) that further enhances sd dominance and that leads to a complete decoupling of the sd subspace in the limit of a large number of bosons. The origin is the repulsion between composite bosons that arises due to Pauli exchange of the constituent nucleons.

To represent this physics, we consider the rational model with bosons up to high spin and with repulsive pairing. Exactly-solvable models of this type can be obtained from a linear combination of the constants of the motion in eq. (4), as in the previous section,

$$H = \sum_l \varepsilon_l K_l^0 + g \sum_{l \neq l'} \frac{\varepsilon_l - \varepsilon_l}{\eta_l - \eta_{l'}} \left[K_l^+ K_{l'}^- - K_l^0 K_{l'}^0 \right] \tag{19}$$

where here l is the boson angular momentum and ε_l are the coefficients of the linear combination of R_l operators. The hamiltonian (19) contains a single-boson term, a repulsive monopole pairing interaction and an attractive monopole-monopole interaction. To obtain the complete set of common eigenstates of the R_l operators and thus of the hamiltonian, it suffices to fix the parameters η_l and the pairing strength g and to look for solutions of the set of non-linear coupled equations (12).

We will present results for two choices of the model parameters. In Case I, we choose $\varepsilon_l = \eta_l = l$, which when inserted in (19) gives rise to a boson hamiltonian with equally spaced single-boson energies and a constant pairing interaction. As discussed before, this hamiltonian favors the scattering of boson pairs to high angular momentum states, a pathology that in fermion problems is often overcome by introducing an energy cutoff. Thus we also consider Case II with $\varepsilon_l = l$ and $\eta_l = l^2$, in which the pair scattering to high spin states has been softened. Once the set of parameters η_l is fixed, we can solve the set of equations (12) for given values of M and g. We can then calculate the occupation probabilities for the various boson states associated with the solution.

For repulsive pairing and a large number of bosons, the system shows a quantum phase transition from an s boson condensate in the weak-pairing limit to a fragmented state in which only the two lowest boson states are macroscopically populated. In our case the two lowest states are precisely the s and d bosons. Quantum phase transitions take place in the thermodynamic limit, where the decoupling is complete. For finite systems the transition is softened and some non-sd bosons persist.

In Figure 4 we show the effect of the boson number on the fragmented phase for the two different cases, assuming a cutoff in boson angular momentum of $L = 12$. In the lower panel, we display the occupation probabilities of the s and d bosons which in the large M limit both go to the well known $O(6)$ value of $1/2$.

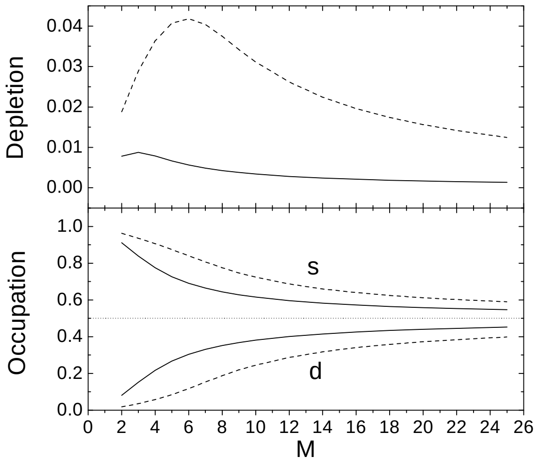

Figure 4. Occupation probabilities as a function of the number of boson pairs M for hamiltonian (19)with $g = 0.5$. The dashed lines refer to Case I and the solid lines to Case II.

In the upper panel, we plot the depletion from the sd subspace, or equivalently the summed occupation probability of bosons with $L > 2$. In Case I, the depletion is small, but non- negligible, for a small number of bosons ($\approx\ 10$) due to the unphysical properties of the constant pairing interaction, but then goes to zero for increasing M values. In Case II, the depletion is always small and goes to zero for moderate values of M.

Knowing the form of the ground state wave function, we can determine why only the s and d boson degrees of freedom survive in the thermodynamic limit. As discussed in ref. [7], specific phase relations must be satisfied by the different degrees of freedom for them to correlate under the influence of repulsive pairing, and these relations can be satisfied by only two at a time. Obviously they will be the lowest two, the s and the d.

Acknowledgments

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