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Deformed Fermion Realization of the sp(4) Algebra and its Application

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1 Introduction

Interest in symplectic groups is related to applications to nuclear structure [1], when the number of particles or couplings between the particles change in a pairwise fashion from one configuration to the next. In particular, $Sp(4)$,which is isomorphic to $O(5)$, has been used to explore pairing correlations in nuclei [2]. The reduction chains to different realizations of the $u(2)$ subalgebra of $sp(4)$ yield a complete classification scheme for the basis states.

Deformed algebras introduce a new degree of freedom, that account for nonlinear effects. Their study can lead to deeper understanding of the physical significance of the deformation. We introduce a q-deformation of the fermions, specific for the applications in the nuclear pairing problem. The deformed $sp_q(4)$ algebra generated by the bilinear products of the deformed creation and annihilation fermion operators is the enveloping algebra of $sp(4)$ and is applied to account for the higher order terms of the pairing interaction. The phenomenological Hamiltonian of a model, which is constructed by the algebraic generators, has $sp_q(4)$ as a dynamical symmetry. The latter is applied to describe the energies of the 0^+ states in fully isovector-paired (pairs with isospin $T = 1$) even-A nuclei, classified in the even representation of the algebra. This application is particularly important for predicting masses of nuclei in a single or multiple light $j-$shells, where pairing correlations play a major role.

2 Generalized Deformed Fermion Realization of sp(4) Algebra

The deformation of the $sp_q(4)$ algebra is introduced in terms of q -deformed creation and annihilation operators $\alpha^\dagger_{m,\sigma}$ and $\alpha_{m,\sigma}$, $(\alpha^\dagger_{m,\sigma})^* = \alpha_{m,\sigma}$, where these operators create (annihilate) a particle of type σ in a state of total angular momentum $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ...$, with projection m along the z axis ($-j \leq m \leq j$).The deformed single-particle operators $\alpha^\dagger_{j,m,\sigma}$ and $\alpha_{j,m,\sigma}$, $(\alpha^\dagger_{j,m,\sigma})^* = \alpha_{j,m,\sigma}$, are the building blocks for the natural expansion to multi-shells dimension [3], which generalizes the fermion realization of the $sp(4)$ algebra to allow the nucleons to occupy a space of several orbits. For a given σ, the dimension of the fermion space is $2\Omega = \sum_j 2\Omega_j = \sum_j (2j+1)$, where the sum is over the number of orbitals. $\alpha^\dagger_{j,m,\sigma}$ and $\alpha_{j,m,\sigma}$, The deformed $sp_q(4)$ is realized in terms of the q-deformed creation and annihilation operators $\alpha^\dagger_{j,m,\sigma}$ and $\alpha_{j,m,\sigma}$, $(\alpha^\dagger_{j,m,\sigma})^* = \alpha_{j,m,\sigma}$, with anticommutation relations defined for every σ, m and j in the form:

$$
\begin{aligned}
&\{\alpha_{j,\sigma,m}, \alpha^\dagger_{j,\sigma,m'}\}_{q^{\pm 1}} = q^{\pm \frac{\tilde{N}_\sigma}{2\Omega}} \delta_{m,m'}, \{\alpha_{j,\sigma,m}, \alpha^\dagger_{j',\sigma',m'}\} = 0, \sigma \neq \sigma', j \neq j', \\
&\{\alpha^\dagger_{j,\sigma,m}, \alpha^\dagger_{j',\sigma',m'}\} = 0, \qquad \{\alpha_{j,\sigma,m}, \alpha_{j',\sigma',m'}\} = 0,
\end{aligned}
\tag{1}
$$

By definition the q-anticommutator is given as $\{A,B\}_k = AB + q^k BA$. The specific for the physical applications property of the introduced deformation (1) is the dependence of the deformed anticommutation relations on the shell dimension and the operators that count the number of particles:

$$
\widetilde{N}_{\pm 1} = \sum_j \sum_{m=-j}^{j} c^\dagger_{m,\pm 1} c_{m,\pm 1}. \tag{2}
$$

in the multi-orbitals. The q-deformed generators of the generalized $Sp_q(4)$ are related to the corresponding single-level generators, given in terms of the deformed operators $\alpha^\dagger_{m,\sigma}$, $\alpha_{m,\sigma}$ for each fixed value of j [4]:

$$
\begin{aligned}
F_{\sigma,\sigma'} &= \frac{1}{\sqrt{2\Omega_j}\sqrt{(1+\delta_{\sigma,\sigma'})}} \sum_{m=-j}^{j} (-1)^{j-m} \alpha^\dagger_{m,\sigma} \alpha^\dagger_{-m,\sigma'} \\
G_{\sigma,\sigma'} &= \frac{1}{\sqrt{2\Omega_j}\sqrt{(1+\delta_{\sigma,\sigma'})}} \sum_{m=-j}^{j} (-1)^{j-m} \alpha_{-m,\sigma} \alpha_{m,\sigma'}
\end{aligned}
\tag{3}
$$

$$
E_{\pm 1,\mp 1} = \frac{1}{\sqrt{2\Omega_j}} \sum_{m=-j}^{j} \alpha^\dagger_{m,\pm 1} \alpha_{\ m,\mp 1}, \tag{4}
$$

in the following way:

$$
\tilde{F}_{\sigma,\sigma'} = \sum_j \sqrt{\frac{\Omega_j}{\Omega}} F_{\sigma,\sigma'},\ \tilde{G}_{\sigma,\sigma'} = \sum_j \sqrt{\frac{\Omega_j}{\Omega}} G_{\sigma,\sigma'}, \tilde{E}_{\sigma,\sigma'} = \sum_j \sqrt{\frac{\Omega_j}{\Omega}} E_{\sigma,\sigma'}. \tag{5}
$$

Like in the "classical" case [5, 6], the operators $F_{\sigma,\sigma'}, (G_{\sigma,\sigma'})$ 3 create (annihilate) a pair of fermions coupled to total angular momentum and parity $J^\pi = 0^+$ and thus constitute boson-like objects. The Cartan subalgebra contains the operators of the number of fermions of each kind, $\widetilde{N}_{\pm 1}(2)$, which remain non-deformed in this realization of $Sp_q(4)$. The ten operators $\tilde{F}_{\sigma,\sigma'}, \tilde{G}_{\sigma,\sigma'}$ and $\tilde{E}_{\sigma,\sigma'}$ and $\tilde{N}_{\pm 1}$ close on the generalized deformed $sp(4)$ algebra, which follows from their commutation relations. For nuclear structure applications we use the set of the commutation relations that is symmetric with respect to the exchange of the deformation parameter $q \leftrightarrow q^{-1}$. For the pairing problem, the role of the $Sp(4) \sim O(5)$ as a dynamical group is revealed through an investigation of its reduction limits. The reduction chains with the corresponding algebraic structures and Casimir invariants of second order can be introduced in the same way as the single-level realization [4]. Table 1 consists of the four different realizations of a two-dimensional unitary $q-$deformed subalgebras $u^\mu_q(2) \supset u^\mu(1) \oplus su^\mu_q(2)$ ($\mu = \{\tau, 0, \pm\}$) and the second order Casimir operators of the respective $su^\mu_q(2)$. The generators of the $u^\mu(1)$ groups are non deformed and they are the first order invariants of the respective $u^\mu_q(2)$. By definition $[X]_k = \frac{q^{kX} - q^{-kX}}{q^k - q^{-k}}$ and $\rho_\pm = (q^{\pm 1} + q^{\pm\frac{1}{2\Omega}})/2$. The corresponding "classical" formulae are restored in the limit when q goes to 1. The deformed analogues of the "classical" pair-creation (annihilation) operators are components of a tensor of first rank $F_{0,\pm 1}$ ($G_{0,\pm 1}$), where $F_{\frac{\sigma+\sigma'}{2}} \equiv F_{\sigma,\sigma'}$ $(G_{\frac{\sigma+\sigma'}{2}} \equiv \; G_{\sigma,\sigma'}), \sigma, \sigma' = \pm 1$, with respect to the $SU^\tau_q(2)$ subgroup.

Table 1. Realizations of the unitary subalgebras of $sp_{(q)}(4)$, $\mu = \{\tau, 0, \pm\}$

$u^\mu(1)$	$su^\mu_q(2)$	$\boldsymbol{C}_2(su^\mu_q(2))$
$N = N_{+1} + N_{-1}$	$T_\pm \equiv E_{\pm 1,\mp 1}$ $T_0 \equiv \tau_0 = \dfrac{N_1 - N_{-1}}{2}$	$2\Omega_j \;\; T_- T_+ + \left[\dfrac{T_0}{2\Omega_j}\right][T_0 + 1]_{\frac{1}{2\Omega_j}}$
τ_0	F_0, G_0 $K_0 \equiv \dfrac{N}{2} - \Omega_j$	$2\Omega_j \;\; G_0 F_0 + \left[\frac{K_0}{2\,\Omega_j}\right][K_0 + 1]_{\frac{1}{2\Omega_j}}$
$N_{\mp 1}$	$F_{\pm 1}, G_{\pm 1}$ $K_{\pm 1} = \dfrac{N_{\pm 1} - \Omega_j}{2}$	$\Omega_j \;\; G_{\pm 1} F_{\pm 1} + \rho_\pm \left[\frac{K_{\pm 1}}{\Omega_j}\right][K_{\pm 1} + 1]_{\frac{1}{\Omega_j}}$

It is interesting to point out that for a single-j orbit the deformed generators do not close within the single-level symplectic algebra, *i.e.* $[T_+, T_-] \neq [2\frac{T_0}{2\Omega}]$, but rather within the generalized $sp(4)$ algebra

$$[T_+, T_-] = [2\frac{\tilde{T}_0}{2\Omega}],\; [K_0, K_0] = [2\frac{\tilde{K}_0}{2\Omega}],\; [F_{\pm 1}, G_{\pm 1}] = \rho_\pm [4\frac{\tilde{K}_{\pm 1}}{2\Omega}].$$

When the index σ is interpreted as defining the nucleon charge ($\sigma = +1$ for protons and $\sigma = -1$ for neutrons), the unitary subalgebra $su^{\tau}(2)$ is associated with the isospin of the valence particles. The $SU^{0}(2)$ limit describes proton-neutron pairs (pn), while the $SU^{\pm}(2)$ limit is related to coupling between identical particles, proton-proton (pp) and neutron-neutron (nn) pairs. The eigenvalues of the Casimir invariant of $SU_q^{+}(2)$ and $SU_q^{-}(2)$ depend on the coefficient $\rho_{\pm}$, which can then be used to distinguish between proton pairs and neutron pairs. In the deformed case (Table 1) the algebra generators retain their "classical" meaning, but they are used not as representing physical observables, but as building blocks of the pairing interactions and their respective basis states.

The deformed fermion operators act in finite deformed spaces $\mathcal{E}_j$, with a vacuum $|0\rangle$ defined by $\alpha_{m,\sigma}|0\rangle = 0$ and $\langle 0|0\rangle = 1$. The q-deformed states are in general different from the classical ones and coincide with them in the limit $q \to 1$. The space $\mathcal{E}_j^{+}$ of fully-paired states is constructed by the pair-creation q-deformed operators $F_{0,\pm 1}$, acting on the vacuum state [7]:

$$|\Omega_j; n_1, n_0, n_{-1})_q = (F_1)^{n_1} (F_0)^{n_0} (F_{-1})^{n_{-1}} |0\rangle \,. \tag{6}$$

where n_1, n_0, n_{-1} are the total number of pairs of each kind, pp, pn, nn, respectively. The basis is obtained by orthonormalization of (6). Within a representation, Ω_j is dropped from the labeling of the states. In the space spanned over the multiple orbitals of dimension 2Ω the basis states are constructed in terms of the generalized deformed pair-creation operators $\tilde{F}_{\sigma}, \sigma = 0, \pm 1$ in the same way as (6). All the formulae, which are derived for a single-j case, have the same content but using the generalized generators (5) and replacing Ω_j by Ω.

In physical applications the quantum numbers that specify the basis states are non-deformed eigenvalues of the operators associated with the "classical" $u^{\mu}(2)$ subalgebras, $\mu = \{0, \pm\}$, corresponding to the different ways of coupling the nucleons. In this way we obtain a full description of the irreducible unitary representations of $U_q(2)$ of four different realizations of $u_q(2)$: $u_q^{\tau}(2),\ u_q^{0}(2)$ and $u_q^{\pm}(2)$.

The basis states $|n_1, n_0, n_{-1})_{(q)}$ give the isovector-paired 0^+ states of a nucleus with $N_+ = 2n_1 + n_0$ valence protons and $N_- = 2n_{-1} + n_0$ valence neutrons. This yields a simultaneous classification of the nuclei in a given j shell and of their corresponding states. The classification scheme is illustrated for the case of $1f_{7/2}$ with $\Omega_{j=7/2} = 4$ (Table 2). The total number of the valence particles, $n = N_+ + N_-$, enumerates the rows and the third projection of the valence isospin, τ_0, enumerates the columns. As a consequence of the charge independence (the exchange $n_1 \leftrightarrow n_{-1}$) and the particle-hole symmetry, the table is symmetric with respect to τ_0 (with the exchange $n_1 \leftrightarrow n_{-1}$), as well as with respect to $n - 2\Omega$ (middle of the shell). According to these symmetries Table 2 can be filled in with nuclei. Isotopes of an element are situated along the right diagonals, isotones – along the left diagonals, and the rows consist of isobars for a given mass number. Hole pair-creation (annihilation) operators can

Table 2. Classification scheme of nuclei, $\Omega_{7/2} = 4$.

n/τ_0	0	-1	-2	-3	-4
0	$\|0,0,0\rangle$ ${}^{40}_{20}$Ca$_{20}$				
2	$\|0,1,0\rangle$ ${}^{42}_{21}$Sc$_{21}$	$\|0,0,1\rangle$ ${}^{42}_{20}$Ca$_{22}$			
4	$\|1,0,1\rangle$ $\|0,2,0\rangle$ ${}^{44}_{22}$Ti$_{22}$	$\|0,1,1\rangle$ ${}^{44}_{21}$Sc$_{23}$	$\|0,0,2\rangle$ ${}^{44}_{20}$Ca$_{24}$		
6	$\|1,1,1\rangle$ $\|0,3,0\rangle$ ${}^{46}_{23}$V$_{23}$	$\|1,0,2\rangle$ $\|0,2,1\rangle$ ${}^{46}_{22}$Ti$_{24}$	$\|0,1,2\rangle$ ${}^{46}_{21}$Sc$_{25}$	$\|0,0,3\rangle$ ${}^{46}_{20}$Ca$_{26}$	
8	$\|2,0,2\rangle$ $\|1,2,1\rangle$ $\|0,4,0\rangle$ ${}^{48}_{24}$Cr$_{24}$	$\|1,1,2\rangle$ $\|0,3,1\rangle$ ${}^{48}_{23}$V$_{25}$	$\|0,2,2\rangle$ $\|1,0,3\rangle$ ${}^{48}_{22}$Ti$_{26}$	$\|0,1,3\rangle$ ${}^{48}_{21}$Sc$_{27}$	$\|0,0,4\rangle$ ${}^{48}_{20}$Ca$_{28}$

be introduced not only for identical particle pairs (pp or nn), but also for pn pairs. This corresponds to a change from the particle to the hole number operator, $N_\pm \rightarrow 2\Omega - N_\pm$ for $N_\pm > \Omega$ and $N \rightarrow 4\Omega - N$ for $N > 2\Omega$.

3 Theoretical Model with $sp(4)$ Dynamical Symmetry

The generalized q-deformed model describes pairing correlations in nuclei with valence nucleons that occupy several orbitals. It can be applied to predict the lowest 0^+ isovector-paired states in the interesting regions of nuclei with a ^{56}Ni core and a ^{100}Sn core. In these regions the observed effects are more fully developed and the available data is richer. They include exotic nuclei without available experimental data like nuclides with relatively large proton excess or with $N \approx Z$. The theoretical results will be tested in the new experimental studies which follow from the recent development of radioactive beam facilities and the studies of astrophysical phenomena.

Here we present the application on the example of the simpler single j- shell model. In analogy with the microscopic "classical" approach [7], the most general Hamiltonian of a system with $Sp_q(4)$ dynamical symmetry, which preserves the total number of particles, can be expressed through the group generators in the following way:

$$
\begin{aligned}
H_q = &-\left(\epsilon_j^q - (\frac{1}{2} - 2\Omega)C_q - \frac{D_q}{4}\right)N - G_q F_0 G_0 \\
&- F_q(F_{+1}G_{+1} + F_{-1}G_{-1}) - \frac{1}{2}E_q\left(\{T_+, T_-\} - \left[\frac{N}{2\Omega}\right]\right) \\
&- 2\Omega C_q \left[\frac{K_0}{2\Omega}\right]\left([K_0+1]_{\frac{1}{2\Omega}} + [K_0-1]_{\frac{1}{2\Omega}}\right) \\
&- \Omega D_q \left[\frac{T_0}{2\Omega}\right]\left([T_0+1]_{\frac{1}{2\Omega}} + [T_0-1]_{\frac{1}{2\Omega}}\right) - O_q,
\end{aligned} \tag{7}
$$

where $\epsilon_j^q > 0$ is the Fermi level of the nuclear system, K_0 is related to N (Table 1), $G_q,\ F_q,\ E_q,\ C_q$ and D_q are constant interaction strength parameters and in general they are different than the non-deformed phenomenological parameters. The constant O_q sets the energy of zero particles to be zero:

$$
O_q = -2\Omega C_q \left[\frac{1}{2}\right]\left([\Omega-1]_{\frac{1}{2\Omega}} + [\Omega+1]_{\frac{1}{2\Omega}}\right).
$$

Expressed in terms of the phenomenological parameters, the q-deformed Hamiltonian is chosen to coincide with a non-deformed one in the limit $q \to 1$, which is appropriate for the description of the pairing interactions in a nuclear system. The basis vectors (6) are eigenstates of the limiting forms of the suitably chosen model Hamiltonian (7). In order to analyze the role of each of the different coupling modes , the Hamiltonian in each limit is expressed through the Casimir invariant of the corresponding $SU(2)$ subgroup and as a result the pairing problem is exactly solvable. For pn-coupling the energy eigenvalue of the non-deformed pairing interaction in the $SU^0(2)$ limit is

$$
\varepsilon_{pn} = \frac{G}{\Omega} n_0 \frac{2\Omega - n + n_0 + 1}{2} = \frac{G}{8\Omega}(n - 2\nu_0)(4\Omega - n - 2\nu_0 + 2). \tag{8}
$$

In the like-particle coupling limit the energy of the non-deformed pairing interaction in the limit $SU^{\pm}(2)$ is

$$
\begin{aligned}
\varepsilon_{pp(nn)} &= \frac{F}{\Omega} n_{\pm 1}(\Omega + n_{\pm 1} - N_{\pm} + 1) \\
&= \frac{F}{4\Omega}(N_{\pm} - \nu_1)(2\Omega - N_{\pm} - \nu_1 + 2).
\end{aligned} \tag{9}
$$

In each limit, $\nu_0 = n_1 + n_{-1}$ and $\nu_1 = n_0$ are the respective seniority quantum numbers that count the number of remaining pairs that can be formed after coupling the fermions in the primary pairing mode and they vary by $\Delta\nu_{0,1} = 2$.

To investigate the influence of the deformation on the pairing interaction, the eigenvalue of the deformed pairing Hamiltonian is expanded in orders of $\varkappa$ ($q = e^{\varkappa}$) in each limit

$$\begin{aligned}\varepsilon^q_{pn} =& G_q \left[\frac{1}{2\Omega}\right] \left[\frac{n-2\nu_0}{2}\right]_{\frac{1}{2\Omega}} \left[\frac{4\Omega - n - 2\nu_0 + 2}{2}\right]_{\frac{1}{2\Omega}} \\ =& \frac{G_q}{G}\varepsilon_{pn}\left\{1 + \varkappa^2 \frac{(n_0^2 - 4\Omega^2 - 1) + \left(\frac{2\Omega}{n_0}\frac{\varepsilon_{pn}}{G}\right)^2}{24\Omega^2} + O(\varkappa^4)\right\},\end{aligned} \tag{10}$$

$$\begin{aligned}\varepsilon^q_{pp(nn)} =& F_q \rho_\pm \left[\frac{1}{\Omega}\right] \left[\frac{N_\pm - \nu_1}{2}\right]_{\frac{1}{\Omega}} \left[\frac{2\Omega - N_\pm - \nu_1 + 2}{2}\right]_{\frac{1}{\Omega}} \\ =& \frac{F_q}{F}\varepsilon_{pp(nn)}\{1 \pm \varkappa \frac{1+2\Omega}{4\Omega} \\ &+ \varkappa^2 \frac{(n_{\pm 1}^2 + \frac{\Omega^2}{2} - \frac{5}{8}) + \left(\frac{\Omega}{n_{\pm 1}}\frac{\varepsilon_{pp(nn)}}{F}\right)^2}{6\Omega^2} + O(\varkappa^3)\},\end{aligned} \tag{11}$$

where the non-deformed energies (8) and (9) are the zeroth order approximation of the corresponding deformed pairing energies. While the proton-neutron interaction is even with respect to the deformation parameter $\varkappa$, the identical particle pairing includes also odd terms due to the coefficient $\rho_\pm$. The expansion of the pairing energy brings into account higher order terms and introduces nonlinearity in the pairing interaction.

4 0^+-State Energies for Even-A Nuclei

The eigenvalues of the Hamiltonian (7) describe nuclear isovector-paired 0^+ state energies. In general, the Hamiltonian is not diagonal in the basis set (Table 2). Linear combinations of the basis vectors describe the spectrum of the relevant states for a given nucleus.

In order to investigate the role of the q-deformation, we perform fitting procedures of the eigenvalues of the deformed Hamiltonian (7) to the experimental energies of the lowest 0^+ isovector-paired $(T = 1)$ states [8, 9]. The extremely good agreement with experiment (small χ) is illustrated in Figure 1 for the relevant energies of nuclei in the $1f_{7/2}$ shell with a $^{40}_{20}$Ca core (Table 2), where $N_\pm = 0, \dots, 8$.

The theory predicted the lowest 0^+ isovector-paired state energy of nuclei with a deviation of at most 0.5% of the energy range considered. It determined the strength of the pairing interaction and estimated the phenomenological deformation parameter $q = 1.114$. The fitting procedure, which yielded $F \neq G$ and $D \neq E/2\Omega$ confirmed also the isospin mixing of the calculated state vectors of the model space ($1f_{7/2}$).

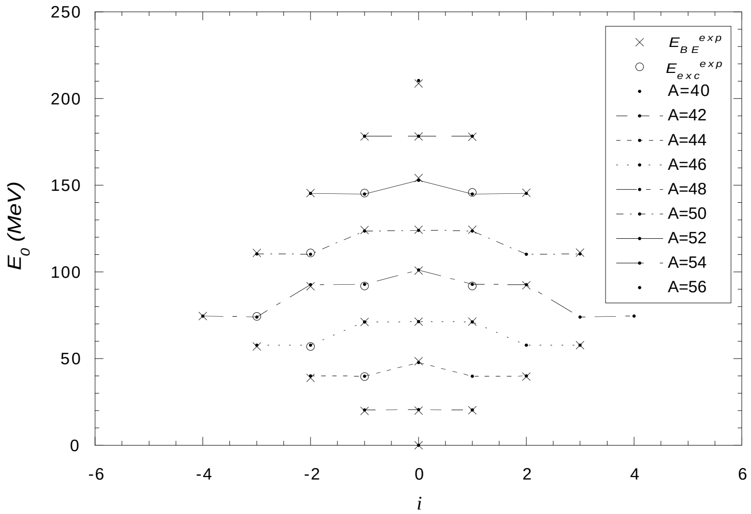

Figure 1. Coulomb corrected 0^+ state energy E_0 vs. i for the isotopes of nuclei with $Z = 20$ to $Z = 28$ in the $1f_{7/2}$ level, $\Omega_{7/2} = 4$. The experimental binding energies E_{BE}^{exp} (symbol "$\times$") are distinguished from the experimental energies of the 0^+ excited states E_{exc}^{exp} (symbol "$\circ$"). Each line connects theoretically predicted energies of an isobar sequence. The nuclei for which experimental data is not available are represented only by their predicted energy.

5 Conclusions

The deformed realization of $sp_q(4)$ is based on the specific q-deformation of a two component Clifford algebra, realized in terms of creation and annihilation fermion operators. The deformed generators of $Sp_q(4)$ close different realizations of the compact $u_q(2)$ subalgebra. Each reduction into compact subalgebras of $sp_q(4)$ provides for a description of a different physical model with different dynamical symmetries. While within a particular deformation scheme the basis states may either be deformed or not, the generators are always deformed as is their action on basis states. With a view towards applications, the additional parameter of the deformation gives in a Hamiltonian theory a dependence of the matrix elements on the $q-$deformation , which does not simply account for one more higher order of a two-body interaction, but it includes all of them through an exponential expansion in parameter $\varkappa$, $q = e^{\varkappa}$. In this way only one parameter, q, can restore the neglected non-linear terms of the residual interaction.

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