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A Chain of Potentials Interpolating between the U(5) and E(5) Symmetries

Dennis Bonatsos[1], **D. Lenis**[1], **N. Minkov**[2], **P. P. Raychev**[2], **and P. A. Terziev**[2]

[1]Institute of Nuclear Physics, N.C.S.R. "Demokritos", GR-15310 Aghia Paraskevi, Attiki, Greece

[2]Institute of Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Sofia 1784, Bulgaria

Abstract.

It is proved that the potentials of the form β^{2n} (with n being integer) provide a "bridge" between the U(5) symmetry of the Bohr Hamiltonian with a harmonic oscillator potential (occuring for $n = 1$) and the E(5) model of Iachello (Bohr Hamiltonian with an infinite well potential, materialized for $n \to \infty$). Parameter-free (up to overall scale factors) predictions for spectra and B(E2) transition rates are given for the potentials β^4, β^6, β^8, corresponding to $R_4 = E(4)/E(2)$ ratios of 2.093, 2.135, 2.157 respectively, compared to the R_4 ratios 2.000 of U(5) and 2.199 of E(5). Hints about nuclei showing this behaviour, as well as about potentials "bridging" the E(5) symmetry with O(6) are briefly discussed. A note about the appearance of Bessel functions in the framework of E(n) symmetries is given as a by-product.

1 Introduction

Models providing parameter-independent predictions for nuclear spectra and electromagnetic transition rates serve as useful benchmarks in nuclear theory. The recently introduced E(5) [1] and X(5) [2] models belong to this category, since their predictions for nuclear spectra (normalized to the excitation energy of the first excited state) and B(E2) transition rates (normalized to the B(E2) transition rate connecting the first excited state to the ground state) do not contain any free parameters. The E(5) model appears to be related to a phase transition from U(5) (vibrational) to O(6) (γ-unstable) nuclei [1], while X(5) is related to a phase transition from U(5) (vibrational) to SU(3) (prolate deformed) nuclei [2]. Both models originate (under certain simplifying assumptions) from the Bohr collective Hamiltonian [3], which is known to possess the U(5) symmetry of the five-dimensional (5-D) harmonic oscillator [4].

In the present paper we study a sequence of potentials building a “bridge” between the U(5) symmetry of the Bohr Hamiltonian and the E(5) model. The potentials are of the form $u_{2n}(\beta) = \beta^{2n}/2$, with n being integer. The Bohr Hamiltonian is obtained for $n = 1$, while E(5) occurs for $n \to \infty$ (in practice $n = 4$ is already quite close to E(5)). Parameter-independent predictions for the spectra and B(E2) values (up to the overall scales mentioned above) are obtained for the potentials β^4, β^6, β^8. In addition to providing a number of models giving predictions directly comparable to experiment, the present sequence of potentials shows the way for approaching the E(5) symmetry starting from U(5) and gives a hint on how to approach the E(5) symmetry starting from O(6).

In Section 2 of the present paper a sequence of potentials providing a “bridge” between the U(5) model of Bohr [3, 4] and the E(5) model of Iachello [1] is introduced. Numerical results for spectra and B(E2) transition rates are shown in Sections 3 and 4 respectively, while Section 5 contains a note on the appearance of Bessel functions in the framework of E(n). Perspectives for further experimental and theoretical work are discussed in Section 6, while in Section 7 the conclusions are summarized.

2 E(5), U(5), and a Sequence of Potentials between Them

The original Bohr Hamiltonian [3] is

$$H = -\frac{\hbar^2}{2B}\Big[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\sum_{k=1,2,3}\frac{Q_k^2}{\sin^2\left(\gamma - \frac{2}{3}\pi k\right)}\Big] + V(\beta,\gamma)\,, \quad (1)$$

where β and γ are the usual collective coordinates describing the shape of the nuclear surface, Q_k ($k = 1, 2, 3$) are the components of angular momentum, and B is the mass parameter.

Assuming that the potential depends only on the variable β, i.e. $V(\beta,\gamma) = U(\beta)$, one can proceed to separation of variables in the standard way [3,5], using the wavefunction

$$\Psi(\beta,\gamma,\theta_i) = f(\beta)\Phi(\gamma,\theta_i)\,, \quad (2)$$

where θ_i $(i = 1, 2, 3)$ are the Euler angles describing the orientation of the deformed nucleus in space.

The equation involving the angles turns out to be

$$\left[-\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma}+\frac{1}{4}\sum_{\kappa}\frac{Q_\kappa^2}{\sin^2(\gamma-\frac{2}{3}\pi\kappa)}\right]\Phi(\gamma,\theta_i)=\Lambda\Phi(\gamma,\theta_i)\ , \quad (3)$$

where $\Lambda = \tau(\tau+3)$ are the eigenvalues of the second order Casimir invariant of SO(5), while $\tau = 0,\ 1,\ 2,\ \ldots$ is the quantum number characterizing the irreducible representations (irreps) of SO(5), called the "seniority" [6]. This equation has been solved by Bes [7].

The "radial" equation

$$\left[-\frac{\hbar^2}{2B}\left(\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta}-\frac{\Lambda}{\beta^2}\right)+U(\beta)\right]f(\beta)=Ef(\beta) \quad (4)$$

can be simplified by introducing [1] reduced energies $\epsilon = \dfrac{2B}{\hbar^2}E$ and reduced potentials $u = \dfrac{2B}{\hbar^2}U$

$$\left[-\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta}+\frac{\Lambda}{\beta^2}+u(\beta)\right]f(\beta)=\epsilon f(\beta)\ , \quad (5)$$

as well as by making the transformation [1] $\phi(\beta)=\beta^{3/2}f(\beta)$, leading to

$$\phi''+\frac{\phi'}{\beta}+\left[\epsilon-u(\beta)-\frac{\left(\tau+\frac{3}{2}\right)^2}{\beta^2}\right]\phi=0\ . \quad (6)$$

For $u(\beta)=\beta^2/2$ one obtains the original solution of Bohr [3], which corresponds to a 5-dimensional (5-D) harmonic oscillator characterized by the symmetry U(5) $\supset$ SO(5) $\supset$ SO(3) $\supset$SO(2) [4], the eigenfunctions being proportional to Laguerre polynomials [8]

$$F_\nu^\tau(\beta)=\left[\frac{2\nu!}{\Gamma\left(\nu+\tau+\frac{5}{2}\right)}\right]^{1/2}\beta^\tau L_\nu^{\tau+3/2}(\beta^2)e^{-\beta^2/2}\ , \quad (7)$$

where $\Gamma(n)$ stands for the Γ-function, and the spectrum having the simple form

$$E_N = N+\frac{5}{2}, \qquad N=2\nu+\tau, \qquad \nu=0,1,2,3,\ldots \quad (8)$$

For $u(\beta)$ being a 5-D infinite well

$$u(\beta)=\begin{cases} 0 & \text{if } \beta\le\beta_W \\ \infty & \text{for } \beta>\beta_W \end{cases} \quad (9)$$

one obtains the E(5) model of Iachello [1], in which the eigenfunctions are Bessel functions $J_{\tau+3/2}(z)$ (with $z = \beta k$, $k = \sqrt{\epsilon}$), while the spectrum is determined by the zeros of the Bessel functions

$$E_{\xi,\tau} = \frac{\hbar^2}{2B} k_{\xi,\tau}^2, \qquad k_{\xi,\tau} = \frac{x_{\xi,\tau}}{\beta_W} \tag{10}$$

where $x_{\xi,\tau}$ is the ξ-th zero of the Bessel function $J_{\tau+3/2}(z)$. The relevant symmetry in this case is E(5)⊃SO(5)⊃SO(3)⊃SO(2), where the Euclidean algebra in 5 dimensions, E(5), is generated by the 5-D momenta π_μ and the 5-D angular momenta $L_{\mu\nu}$, while SO(5) is generated by the $L_{\mu\nu}$ alone [2]. τ, L, and M are the quantum numbers characterizing the irreps of SO(5), SO(3), and SO(2) respectively. The values of angular momentum L contained in each irrep of SO(5) (i.e. for each value of τ) are given by the algorithm [9]

$$\tau = 3\nu_\Delta + \lambda, \qquad \nu_\Delta = 0, 1, \dots\ , \tag{11}$$

$$L = \lambda, \lambda+1, \dots, 2\lambda-2, 2\lambda \tag{12}$$

(with $2\lambda - 1$ missing), where ν_Δ is the missing quantum number in the reduction SO(5) ⊃ SO(3), and are listed in Table 1.

Table 1. Quantum numbers appearing in the SO(5)⊃SO(3) reduction [9], occuring from Eqs. (11) and (12).

τ	ν_Δ	λ	L
0	0	0	0
1	0	1	2
2	0	2	4,2
3	0	3	6,4,3
3	1	0	0
4	0	4	8,6,5,4
4	1	1	2
5	0	5	10,8,7,6,5
5	1	2	4,2
6	0	6	12,10,9,8,7,6
6	1	3	6,4,3
6	2	0	0

The spectra of the $u(\beta) = \beta^2/2$ potential and of the E(5) model become directly comparable by establishing the formal correspondence

$$\nu = \xi - 1 \; . \tag{13}$$

It should be emphasized that the quantum numbers appearing in Eq. (13) have different origins, ν being an oscillator quantum number labeling the number of zeros of a Laguerre polynomial, while ξ is labeling the order of a zero of a Bessel function. Eq. (13) establishes a formal one-to-one correspondence between the states in the two spectra and allows one to continue using for the states the notation $L_{\xi,\tau}$ (where L is the angular momentum), as in Ref. [1], although a notation $L_{\nu,\tau}$ would have been equally appropriate. The ground state band corresponds to $\xi = 1$ (or, equivalently, $\nu = 0$).

The two cases mentioned above are the only ones in which Eq. (6) is exactly soluble, giving spectra characterized by $R_4 = E(4)/E(2)$ ratios 2.00 and 2.20 respectively. However, the numerical solution of Eq. (6) for potentials other than the ones mentioned above is a straightforward task [10], in which one uses the chain U(5)⊃SO(5)⊃SO(3)⊃SO(2) for the classification of the states.

Not all potentials can be used in Eq. (6), though, since they have to obey the restrictions imposed by the 24 transformations mentioned in [3] and listed explicitly in [11]. These restrictions allow the presence of even powers of β in the potentials, while odd powers of β should be accompanied by $\cos 3\gamma$ [12].

A particularly interesting sequence of potentials is given by

$$u_{2n}(\beta) = \frac{\beta^{2n}}{2} \; , \tag{14}$$

with n being an integer. For $n = 1$ the Bohr case (U(5)) is obtained, while for $n \to \infty$ the infinite well of E(5) is obtained [13]. Therefore this sequence of potentials provides a “bridge” between the U(5) symmetry and the E(5) model, using their common SO(5)⊃SO(3) chain of subalgebras for the classification of the spectra.

3 Spectra

Numerical results for the spectra of the β^4, β^6, and β^8 potentials have been obtained through two different methods. In one approach, the representation of the position and momentum operators in matrix form [14] has been used, while in the other the direct integration method [15] has been applied. In the latter, the differential equation is solved for each value of $\tau = 0, 1, 2, \ldots$ separately, the successive eigenvalues for each value of τ labeled by $\xi = 1, 2, 3, \ldots$ (or, equivalently, by $\nu = 0, 1, 2, \ldots$). The two methods give results mutually consistent, the second one appearing of more general applicability. The results are shown in Table 2, where excitation energies relative to the ground state, normalized to the excitation energy of the first excited state, are exhibited.

Table 2. Spectra of the E(5)-β^4, E(5)-β^6, and E(5)-β^8 models, compared to the predictions of the U(5) (Eq. (8)) and E(5) (Eq. (10)) models. For each value of τ, only the maximum value of L occuring for it, L_{max}, is reported. The rest of the allowed values of L for each value of τ, indicating states having the same energy as the state with L_{max}, can be read from Table 1.

band	τ	L_{max}	U(5)	E(5)-β^4	E(5)-β^6	E(5)-β^8	E(5)
$\xi=1$							
	0	0	0.000	0.000	0.000	0.000	0.000
	1	2	1.000	1.000	1.000	1.000	1.000
	2	4	2.000	2.093	2.135	2.157	2.199
	3	6	3.000	3.265	3.391	3.459	3.590
	4	8	4.000	4.508	4.757	4.894	5.169
	5	10	5.000	5.813	6.225	6.456	6.934
	6	12	6.000	7.176	7.788	8.138	8.881
	7	14	7.000	8.592	9.442	9.935	11.009
	8	16	8.000	10.057	11.180	11.841	13.316
	9	18	9.000	11.569	13.000	13.854	15.799
	10	20	10.000	13.124	14.898	15.968	18.459
	11	22	11.000	14.720	16.871	18.182	21.294
	12	24	12.000	16.355	18.916	20.492	24.302
	13	26	13.000	18.028	21.031	22.896	27.484
	14	28	14.000	19.737	23.213	25.391	30.837
	15	30	15.000	21.480	25.460	27.975	34.363
$\xi=2$							
	0	0	2.000	2.390	2.619	2.756	3.031
	1	2	3.000	3.625	4.012	4.255	4.800
	2	4	4.000	4.918	5.499	5.874	6.780
	3	6	5.000	6.266	7.075	7.607	8.967
	4	8	6.000	7.666	8.738	9.450	11.357
	5	10	7.000	9.115	10.483	11.400	13.945
$\xi=3$							
	0	0	4.000	5.153	5.887	6.364	7.577
	1	2	5.000	6.563	7.588	8.269	10.107
	2	4	6.000	8.015	9.363	10.274	12.854
	3	6	7.000	9.509	11.213	12.379	15.814
	4	8	8.000	11.043	13.134	14.580	18.983
	5	10	9.000	12.617	15.125	16.875	22.359
$\xi=4$							
	0	0	6.000	8.213	9.698	10.707	13.639
	1	2	7.000	9.764	11.661	12.966	16.928
	2	4	8.000	11.349	13.687	15.316	20.436
	3	6	9.000	12.967	15.776	17.753	24.161
	4	8	10.000	14.619	17.928	20.278	28.100
	5	10	11.000	16.304	20.141	22.888	32.250

Table 3. Same as Table 2, but for spectra of the potentials of Eq. (15) for different values of the control parameter η, compared to the predictions of the U(5) ($\eta = 0$, Eq. (8)) and E(5) (Eq. (10) models.

		η	0	1/4	1/2	3/4	1	E(5)
band	τ	L_{max}						
$\xi = 1$								
	0	0	0.000	0.000	0.000	0.000	0.000	0.000
	1	2	1.000	1.000	1.000	1.000	1.000	1.000
	2	4	2.000	2.063	2.093	2.114	2.130	2.199
	3	6	3.000	3.183	3.265	3.323	3.368	3.590
	4	8	4.000	4.353	4.508	4.615	4.700	5.169
	5	10	5.000	5.569	5.813	5.982	6.115	6.934
	6	12	6.000	6.828	7.176	7.415	7.605	8.881
	7	14	7.000	8.127	8.592	8.911	9.164	11.009
	8	16	8.000	9.462	10.057	10.464	10.786	13.316
	9	18	9.000	10.833	11.569	12.071	12.467	15.799
	10	20	10.000	12.238	13.124	13.727	14.204	18.459
	11	22	11.000	13.674	14.720	15.432	15.994	21.294
	12	24	12.000	15.140	16.355	17.181	17.833	24.302
	13	26	13.000	16.636	18.028	18.973	19.719	27.484
	14	28	14.000	18.159	19.737	20.807	21.651	30.837
	15	30	15.000	19.709	21.480	22.679	23.625	34.363
$\xi = 2$								
	0	0	2.000	2.251	2.390	2.498	2.590	3.031
	1	2	3.000	3.419	3.625	3.776	3.902	4.800
	2	4	4.000	4.629	4.918	5.124	5.292	6.780
	3	6	5.000	5.881	6.266	6.537	6.756	8.967
	4	8	6.000	7.171	7.666	8.011	8.287	11.357
	5	10	7.000	8.497	9.115	9.542	9.883	13.945
$\xi = 3$								
	0	0	4.000	4.785	5.153	5.419	5.639	7.577
	1	2	5.000	6.082	6.563	6.905	7.182	10.107
	2	4	6.000	7.412	8.015	8.438	8.779	12.854
	3	6	7.000	8.774	9.509	10.020	10.430	15.814
	4	8	8.000	10.166	11.043	11.649	11.133	18.983
	5	10	9.000	11.589	12.617	13.324	13.887	22.359
$\xi = 4$								
	0	0	6.000	7.548	8.213	8.681	9.061	13.639
	1	2	7.000	8.951	9.764	10.331	10.788	16.928
	2	4	8.000	10.382	11.349	12.019	12.556	20.436
	3	6	9.000	11.839	12.967	13.745	14.366	24.161
	4	8	10.000	13.322	14.619	15.509	16.218	28.100
	5	10	11.000	14.831	16.304	17.311	18.111	32.250

In Table 2 the labels E(5)-β^4, E(5)-β^6, E(5)-β^8 have been used for the above-mentioned potentials, their meaning being that E(5)-β^{2n} corresponds to the potential $\beta^{2n}/2$ plugged in the differential equation obtained in the framework of the E(5) model. In this notation E(5)-β^2 coincides with the original U(5) model of Bohr [3], while E(5)-β^{2n} with $n \to \infty$ is simply the original E(5) model [1].

From Table 2 it is clear that in all bands and for all values of the angular momentum, L, the potentials β^4, β^6, β^8 gradually lead from the U(5) case to the E(5) results in a smooth way.

It is instructive to compare the results obtained with the potentials of Eq. (14) to the ones provided by the potentials [1, 16]

$$u(\beta) = \frac{1}{2}(1-\eta)\beta^2 + \frac{\eta}{4}(1-\beta^2)^2 , \tag{15}$$

where η is a control parameter. Results for the spectra of these potentials (for $\eta = 1/4$, 1/2, 3/4, 1) are shown in Table 3, while for $\eta = 0$ it is clear that the Bohr U(5) case is reproduced. The following observations can be made:

1) For $\eta = 1/2$ the results coincide with these of E(5)-β^4, as expected, since for $\eta = 1/2$ Eq. (15) gives $u(\beta) = (\beta^4 + 1)/8$, while in Tables 2 and 3 excitation energies relative to the ground state and normalized to the excitation energy of the first excited state are shown.

2) Giving to the control parameter η the values 0, 1/4, 1/2, 3/4, 1, one obtains spectra characterized by R_4 ratios 2.00, 2.06, 2.09, 2.11, 2.13 respectively. Thus one cannot reach $R_4 = 2.20$, which is a hallmark of E(5). For approaching $R_4 = 2.20$ one needs higher powers of β, as seen in Table 2 (β^8 already gives $R_4 = 2.16$, while higher powers of β will provide R_4 values even closer to 2.20).

3) It has been noticed in [1] that the potential should exhibit a flat behaviour when the system undergoes a phase transition. The sequence of potentials given in Eq. (14) is indeed a series of gradually flatter (with increasing n) potentials, giving the infinite well potential of E(5) as a limiting case. These potentials therefore do provide a complete “bridge” between U(5) and E(5).

4 B(E2) Transition Rates

In nuclear structure it is well known that electromagnetic transition rates are quantities sensitive to the details of the underlying microscopic structure, as well as to details of the theoretical models, much more than the corresponding spectra. It is therefore a must to calculate B(E2) ratios (normalized to B(E2:$2_1^+ \to 0_1^+$)=100) for the potentials of Eq. (14).

The quadrupole operator has the form [5]

$$T_\mu^{(E2)} = t\alpha_\mu = \\ = t\beta \left[\mathcal{D}^{(2)}_{\mu,0}(\theta_i) \cos\gamma + \frac{1}{\sqrt{2}} (\mathcal{D}^{(2)}_{\mu,2}(\theta_i) + \mathcal{D}^{(2)}_{\mu,-2}(\theta_i)) \sin\gamma \right] , \quad (16)$$

where t is a scale factor and $\mathcal{D}(\theta_i)$ denote Wigner functions of the Euler angles, while the B(E2) transition rates are given by

$$B(E2; \varrho_i L_i \to \varrho_f L_f) = \frac{1}{2L_i+1} |\langle \varrho_f L_f || T^{(E2)} || \varrho_i L_i \rangle|^2 = \\ = \frac{2L_f+1}{2L_i+1} B(E2; \varrho_f L_f \to \varrho_i L_i) , \quad (17)$$

where by ϱ quantum numbers other than the angular momentum L are denoted.

The states with $\nu_\Delta = 0$ and $L = 2\tau$ can be written in the form dictated by Eq. (2)

$$|\xi, \tau, \nu_\Delta = 0, L = 2\tau, M = L\rangle = f_{\xi\tau}(\beta)\, \Phi^{\tau,\, \nu_\Delta=0}_{L=2\tau,\, M=L}(\gamma, \theta_i) = \\ = f_{\xi\tau}(\beta)\, \phi_\tau(\gamma, \theta_i) , \quad (18)$$

where the functions $\phi_\tau(\gamma, \theta_i)$ have the form [7]

$$\phi_\tau(\gamma, \theta_i) = \frac{1}{\sqrt{A_\tau}} \left(\frac{\alpha_2}{\beta} \right)^\tau , \quad (19)$$

with the normalization factor

$$A_\tau = \frac{\tau!}{(2\tau+3)!!} (4\pi)^2 \quad (20)$$

determined from the normalization condition

$$\int_{\gamma=0}^{\frac{\pi}{3}} \int \phi^*_\tau(\gamma, \theta_i)\, \phi_\tau(\gamma, \theta_i)\, \sin 3\gamma\, \sin\theta_2\, d\gamma\, d\theta_1\, d\theta_2\, d\theta_3 = 1 . \quad (21)$$

From Eqs. (17) and (19) one obtains

$$B(E2; L_{\xi,\tau} \to (L+2)_{\xi',\tau+1}) = \\ = \frac{(\tau+1)(4\tau+5)}{(2\tau+5)(4\tau+1)} t^2 I^2_{\xi',\tau+1;\, \xi,\tau}; \quad L = 2\tau , \quad (22)$$

$$B(E2;(L+2)_{\xi',\tau+1}\to L_{\xi,\tau})=\frac{\tau+1}{2\tau+5}\,t^2\,I^2_{\xi',\tau+1;\,\xi,\tau};\quad L=2\tau\ , \tag{23}$$

where

$$I_{\xi',\tau+1;\,\xi,\tau}=\int_0^\infty \beta f_{\xi'\,\tau+1}(\beta)\,f_{\xi\tau}(\beta)\beta^4 d\beta\ . \tag{24}$$

In the special case of the potential being a 5-D infinite well the eigenfunctions are

$$f_{\xi\tau}(\beta)=\frac{1}{\sqrt{C_{\xi,\tau}}}\,\beta^{-3/2}J_{\tau+3/2}\Big(x_{\xi,\tau}\,\frac{\beta}{\beta_W}\Big)\ , \tag{25}$$

with

$$C_{\xi,\tau}=\frac{\beta_W^2}{2}J^2_{\tau+5/2}(x_{\xi,\tau})\ , \tag{26}$$

where $x_{\xi,\tau}$ is the ξ-th zero of the Bessel function $J_{\tau+3/2}(z)$, while the constants $C_{\xi,\tau}$ are obtained from the normalization condition

$$\int_0^{\beta_W} f^2_{\xi\tau}(\beta)\,\beta^4 d\beta=1\ . \tag{27}$$

In this case the integrals of Eq. (24) take the form

$$\begin{aligned}I_{\xi',\tau+1;\,\xi,\tau}&=\int_0^{\beta_W}\beta f_{\xi'\,\tau+1}(\beta)\,f_{\xi\tau}(\beta)\beta^4 d\beta=\\&=(C_{\xi',\tau+1}C_{\xi,\tau})^{-1/2}\beta_W^3\int_0^1 z^2 J_{\tau+5/2}(x_{\xi',\tau+1}\,z)\,J_{\tau+3/2}(x_{\xi,\tau}\,z)\,dz\ .\end{aligned} \tag{28}$$

The results of the calculations for intraband transitions are shown in Table 4, while interband transitions are shown in Table 5. In all cases a smooth evolution from U(5) to E(5) is seen. Furthermore, the results are in agreement to general qualitative expectations: the more collective the nucleus, the less rapid the increase (with increasing initial angular momentum) of the B(E2)s within the ground state band should be (in the absence of bandcrossings). Indeed the most rapid increase is seen in the case of U(5), while the slowest increase is observed in the case of E(5). The E(5) results reported in Tables 4 and 5 are in good agreement with the results given in Ref. [17].

Table 4. Intraband B(E2) transition rates for the E(5)-β^4, E(5)-β^6, and E(5)-β^8 models, compared to the predictions of the U(5) and E(5) models. See Section 4 for details.

bands	$(L_{\xi,\tau})_i$	$(L_{\xi,\tau})_f$	U(5)	E(5)-β^4	E(5)-β^6	E(5)-β^8	E(5)
$(\xi=1)\rightarrow(\xi=1)$	$2_{1,1}$	$0_{1,0}$	100.00	100.00	100.00	100.00	100.00
	$4_{1,2}$	$2_{1,1}$	200.00	183.20	176.60	173.32	167.40
	$6_{1,3}$	$4_{1,2}$	300.00	256.37	239.80	231.64	216.88
	$8_{1,4}$	$6_{1,3}$	400.00	322.73	294.27	280.39	255.20
	$10_{1,5}$	$8_{1,4}$	500.00	384.12	342.57	322.51	286.01
	$12_{1,6}$	$10_{1,5}$	600.00	441.65	386.26	359.74	311.47
	$14_{1,7}$	$12_{1,6}$	700.00	496.11	426.36	393.25	332.95
	$16_{1,8}$	$14_{1,7}$	800.00	548.02	463.57	423.80	351.39
	$18_{1,9}$	$16_{1,8}$	900.00	597.78	498.40	451.94	367.44
	$20_{1,10}$	$18_{1,9}$	1000.00	645.69	531.23	478.10	381.56
	$22_{1,11}$	$20_{1,10}$	1100.00	692.00	562.35	502.58	394.10
	$24_{1,12}$	$22_{1,11}$	1200.00	736.89	592.00	525.62	405.34
	$26_{1,13}$	$24_{1,12}$	1300.00	780.52	620.35	547.41	415.48
	$28_{1,14}$	$26_{1,13}$	1400.00	823.01	647.55	568.12	424.68
	$30_{1,15}$	$28_{1,14}$	1500.00	864.47	673.73	587.86	433.09
$(\xi=2)\rightarrow(\xi=2)$	$2_{2,1}$	$0_{2,0}$	140.00	112.64	98.97	91.24	75.22
	$4_{2,2}$	$2_{2,1}$	257.14	197.92	170.97	156.06	124.32
	$6_{2,3}$	$4_{2,2}$	366.67	271.04	230.57	208.71	161.52
	$8_{2,4}$	$6_{2,3}$	472.73	336.84	282.53	253.85	191.58
	$10_{2,5}$	$8_{2,4}$	576.92	397.56	329.12	293.70	216.77
$(\xi=3)\rightarrow(\xi=3)$	$2_{3,1}$	$0_{3,0}$	180.00	126.58	103.69	91.64	65.73
	$4_{3,2}$	$2_{3,1}$	314.29	214.91	173.97	152.67	106.63
	$6_{3,3}$	$4_{3,2}$	433.33	288.38	230.96	201.40	137.44
	$8_{3,4}$	$6_{3,3}$	545.45	353.71	280.48	243.22	162.57
	$10_{3,5}$	$8_{3,4}$	653.85	413.72	325.01	280.39	183.95
$(\xi=4)\rightarrow(\xi=4)$	$2_{4,1}$	$0_{4,0}$	220.00	140.44	109.56	94.03	60.68
	$4_{4,2}$	$2_{4,1}$	371.43	232.42	179.66	153.33	96.89
	$6_{4,3}$	$4_{4,2}$	500.00	306.70	235.08	199.63	123.79
	$8_{4,4}$	$6_{4,3}$	618.18	371.85	282.79	239.04	145.70
	$10_{4,5}$	$8_{4,4}$	730.77	431.31	325.60	274.06	164.42

Table 5. Same as Table 4, but for interband B(E2) transitions.

bands	$(L_{\xi,\tau})_i$	$(L_{\xi,\tau})_f$	U(5)	E(5)-β^4	E(5)-β^6	E(5)-β^8	E(5)
$(\xi=2)\to(\xi=1)$	$0_{2,0}$	$2_{1,1}$	200.00	141.77	118.98	107.57	86.79
	$2_{2,1}$	$4_{1,2}$	102.86	66.10	52.62	46.00	33.82
	$2_{2,1}$	$0_{1,0}$	0.00	0.16	0.30	0.38	0.47
	$4_{2,2}$	$6_{1,3}$	96.30	57.33	43.78	37.263	25.17
	$4_{2,2}$	$2_{1,1}$	0.00	0.24	0.45	0.56	0.69
	$6_{2,3}$	$8_{1,4}$	95.11	53.20	39.26	32.68	20.44
	$6_{2,3}$	$4_{1,2}$	0.00	0.28	0.52	0.65	0.79
$(\xi=3)\to(\xi=2)$	$0_{3,0}$	$2_{2,1}$	400.00	257.90	205.27	178.52	123.22
	$2_{3,1}$	$4_{2,2}$	205.71	123.14	94.54	80.50	51.57
	$2_{3,1}$	$0_{2,0}$	0.00	0.22	0.38	0.46	0.54
	$4_{3,2}$	$6_{2,3}$	192.59	108.39	80.68	67.46	40.44
	$4_{3,2}$	$2_{2,1}$	0.00	0.34	0.58	0.69	0.79
	$6_{3,3}$	$8_{2,4}$	190.21	101.58	73.59	60.54	34.16
	$6_{3,3}$	$4_{2,2}$	0.00	0.42	0.71	0.84	0.92
$(\xi=4)\to(\xi=3)$	$0_{4,0}$	$2_{3,1}$	600.00	358.53	273.82	232.05	144.02
	$2_{4,1}$	$4_{3,2}$	308.57	173.79	129.12	107.67	62.88
	$2_{4,1}$	$0_{3,0}$	0.00	0.26	0.43	0.51	0.56
	$4_{4,2}$	$6_{3,3}$	288.89	154.60	112.08	92.13	50.93
	$4_{4,2}$	$2_{3,1}$	0.00	0.41	0.66	0.77	0.81
	$6_{4,3}$	$8_{3,4}$	285.31	145.99	103.53	84.01	44.16
	$6_{4,3}$	$4_{3,2}$	0.00	0.51	0.82	0.94	0.96

5 A Note on E(n) and Bessel Functions

Concerning the appearance of Bessel functions in the case of E(5), the following mathematical remarks can be made in the general case of the Euclidean algebra in n dimensions, E(n), which is the semidirect sum [18] of the algebra T_n of translations in n dimensions, generated by the momenta

$$P_j = -i\frac{\partial}{\partial x_j} \ , \tag{29}$$

and the SO(n) algebra of rotations in n dimensions, generated by the angular momenta

$$L_{jk} = -i\left(x_j\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial x_j}\right) \ , \tag{30}$$

symbolically written as E(n)=$\mathrm{T_n} \oplus_s$ SO(n) [19]. The generators of E(n) satisfy the commutation relations

$$[P_i, P_j] = 0, \qquad [P_i, L_{jk}] = i(\delta_{ik}P_j - \delta_{ij}P_k) \ , \tag{31}$$

$$[L_{ij}, L_{kl}] = i(\delta_{ik}L_{jl} + \delta_{jl}L_{ik} - \delta_{il}L_{jk} - \delta_{jk}L_{il}) \ . \tag{32}$$

From these commutation relations one can see that the square of the total momentum, P^2, is a second order Casimir operator of the algebra, while the eigenfunctions of this operator satisfy the equation

$$\left(-\frac{1}{r^{n-1}}\frac{\partial}{\partial r}r^{n-1}\frac{\partial}{\partial r}+\frac{\omega(\omega+n-2)}{r^2}\right)F(r)=k^2F(r)\ , \tag{33}$$

in the left hand side of which the eigenvalues of the Casimir operator of SO(n), $\omega(\omega+n-2)$ appear [8]. Putting

$$F(r)=r^{(2-n)/2}f(r)\ , \tag{34}$$

and

$$\nu=\omega+\frac{n-2}{2}\ , \tag{35}$$

Eq. (33) is brought into the form

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+k^2-\frac{\nu^2}{r^2}\right)f(r)=0\ , \tag{36}$$

the eigenfunctions of which are the Bessel functions $f(r)=J_\nu(kr)$ [20]. We see therefore that the Bessel functions appear in general in this type of problems when the potential is vanishing, so that only the kinetic energy term appears in the Hamiltonian.

6 Perspectives

It is interesting to examine if there is any experimental evidence supporting the E(5)-β^{2n} predictions. It is clear that the first regions to be considered are the ones around the nuclei which have been identified as good candidates for E(5), i.e. ^{134}Ba [21], ^{104}Ru [22], ^{102}Pd [23]. A very preliminary search indicates that ^{98}Ru [24] can be a candidate for E(5)-β^6, while ^{100}Pd [25] can be a candidate for E(5)-β^4. Existing data for the ground state bands of these nuclei are compared to the theoretical predictions in Table 6. However, much more detailed information on the spectra and B(E2) transitions of these nuclei are required before final conclusions can be reached.

Concerning future theoretical work, at least two directions open up:

1) One should study a similar sequence of potentials serving as a "bridge" between U(5) and X(5) [2]. A relevant report is given in these proceedings [26].

2) One should try to find a sequence of potentials interpolating between O(6) and E(5), as well as between SU(3) and X(5). In other words, one should try to approach E(5) and X(5) "from the other side". From the classical

limit of the O(6) and SU(3) symmetries of the Interacting Boson Model [9] it is clear that for this purpose potentials with a minimum at $\beta \neq 0$ should be considered, the Davidson-like potentials [27]

$$u_{2n}^D(\beta) = \beta^{2n} + \frac{\beta_0^{4n}}{\beta^{2n}} \tag{37}$$

being strong candidates. The Davidson potential, corresponding to $n = 1$, is known to be exactly soluble [27, 28].

Work in these directions is in progress.

Table 6. Experimental spectra of the ground state bands of ^{100}Pd [25] and ^{98}Ru [24], compared to the predictions of the E(5)-β^4 and E(5)-β^6 models respectively.

L	^{100}Pd	E(5)-β^4	^{98}Ru	E(5)-β^6
2	1.000	1.000	1.000	1.000
4	2.128	2.093	2.142	2.135
6	3.290	3.265	3.406	3.391
8	4.489	4.508	4.792	4.757
10	5.814	5.813	6.091	6.225
12	7.154	7.176		7.788
14	8.574	8.592		9.442
16	10.425	10.057		11.180

7 Conclusion

It has been proved that the potentials β^{2n} (with n being integer) provide a complete "bridge" between the U(5) symmetry of the Bohr Hamiltonian with a harmonic oscillator potential (occuring for $n = 1$) and the E(5) model of Iachello, which is obtained from the Bohr Hamiltonian when an infinite well potential is plugged in it (materialized for $n \to \infty$). Parameter-free (up to overall scale factors) predictions for spectra and B(E2) transition rates have been given for the potentials β^4, β^6, β^8, called the E(5)-β^4, E(5)-β^6, and E(5)-β^8 models, respectively. Hints about nuclei showing this behaviour, as well as about potentials approaching E(5) "from the other side" (i.e. providing a "bridge" between O(6) and E(5)) have been briefly discussed. A mathematical note on the appearance of Bessel functions in the framework of E(n) models has been given as a by-product.

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