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The Most Hidden Symmetry and Nuclear Clusterization

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Abstract.

Different symmetries are discussed in the sequence of increasing generality from the well-known geometrical symmetry up to the effective symmetry. Simple applications in the field of nuclear clusterization are mentioned.

1 Introduction

A symmetry is usually called hidden symmetry, when it refers to a larger invariance group, than the one obvious for the physical system in question [1]. Text-book examples are the o(4) symmetry of the Kepler-problem, and the u(3) symmetry of the three-dimensional harmonic oscillator problem. (Both of them are larger, than the obvious so(3) group of the three-dimensional rotations.) They are also called dynamical symmetries [2], because they involve not only geometrical transformations of the coordinates, but more complex transformations of the dynamical variables as well.

In this contribution we discuss some well-known, hidden, and even more hidden symmetries of quantum mechanics (more hidden than those defined in [1]), which find their applications in nuclear physics.

The structure of the paper is as follows. In Section 2 (the main part of the paper) a step-by-step generalization of the concept of symmetry is described, starting from the simple geometrical symmetry, and ending up with the concept of effective symmetry. The latter one can be considered (in some sense) as the most hidden symmetry of quantum mechanics. The symmetry of a quantum mechanical system is expressed in terms of the symmetry properties of the Hamiltonian operator and its eigenvectors. The geometrical symmetry is characterised by a situation, in which the Hamiltonian, its kinetic and potential energy parts, as well as its eigenvectors have good symmetries. The effective symmetry represents a situation, in which neither the Hamiltonian, nor its eigenvectors are symmetric. Yet the symmetry is present, and it has important physical consequences.

Section 3 describes very briefly a specific effective symmetry, namely the effective su(3) symmetry of atomic nuclei. In section 4 simple applications of the real and effective su(3) symmetry are mentioned in relation with the clusterization of the nuclei. In the final section some conclusions are drawn.

2 Hierarchy of Hidden Symmetries

2.1 Obvious (Geometrical) Symmetry

First we consider the familiar example of the rotational (so(3)) invariance in order to establish our vocabulary, then the physical consequences of this symmetry are listed, and finally we reformulate the basic statements in general terms.

2.1.1 A Well-Known Example

A quantum mechanical system, governed by a Hamiltonian

$$H = T + V \ ,$$

is said to have an exact rotational (geometrical) symmetry if both the total Hamiltonian, and its kinetic and potential energy parts commute with the operators of the angular momentum algebra:

$$[H, J_i] = [T, J_i] = [V, J_i] = 0, \ i = +, 0, - \ ,$$
$$[J_+, J_-] = 2J_0, \ [J_0, J_\pm] = \pm J_\pm \ .$$

2.1.2 Consequences

The exact (rotational) geometrical symmetry has several important consequences.

i) The existence of good quantum numbers: $J, \ M$. ($J(J+1)$ and M are the eigenvalues of the operators J^2 and J_0.) The eigenvectors of a rotationally invariant Hamiltonian can be labelled by these quantum numbers: $|\psi_{JM}\rangle$. In mathematical terms they are indices of the irreducible representations (irreps) of the so(3) and its so(2) subalgebras, respectively. The existence of good quantum numbers provide us with the possibility of defining selection rules (see section 4).

ii) Multiplet structure: states with $M = J, J-1, ..., -(J-1), -J$ form a multiplet.

iii) The $|\psi_{JM}\rangle$ eigenvectors of the Hamiltonian transform according to a simple and algebraically well-defined way (like the basis vectors of the irreducible representations of so(3)) :

$$J_{\pm}|\psi_{JM}\rangle = (J(J+1) - M(M\pm 1))^{\frac{1}{2}}|\psi_{JM\pm 1}\rangle \ , \quad (1)$$
$$J_0|\psi_{JM}\rangle = M|\psi_{JM}\rangle. \quad (2)$$

In short this situation is described by saying that the eigenvectors are symmetric.

iv) The members of a multiplet are degenerate.

v) If H has a symmetry so(3) and in addition it has an so(3) algebraic structure (see the next subsection), then the eigenvalue-problem of H has an analytical solution. E.g. when H is expressed in terms of the invariant (or Casimir) operator J^2 of the so(3) algebra:

$$H = aJ^2 \ ,$$

then its eigenvalues are:

$$E = aJ(J+1) \ .$$

It is worth stressing that some of the consequences i)–iv) apply even if the Hamiltonian is not given explicitly, only its symmetry is known.

2.1.3 Hamiltonian with an Algebraic Structure

H has an so(3) algebraic structure if H can be expressed in terms of the angular momentum operators.

It should be emphasised that the algebraic structure of H and the symmetry of H are two different concepts. E.g.

$$H = av^2 + b\frac{1}{r}$$

(where v stands for velocity) has an so(3) symmetry but does not have an so(3) algebraic structure. On the other hand

$$H = aJ^2$$

has an so(3) symmetry and an so(3) algebraic structure as well.

2.1.4 Matrix Representation

From Eqs. (1) and (2) we obtain the matrix representation of the angular momentum operators:

$$\langle \psi_{JM}|J_{\pm}|\psi_{JM'}\rangle = (J(J+1) - M'(M' \pm 1))^{\frac{1}{2}} \delta_{M,M'\pm 1} \ , \qquad (3)$$
$$\langle \psi_{JM}|J_0|\psi_{JM'}\rangle = M\delta_{M,M'} \ . \qquad (4)$$

2.1.5 In General

A quantum mechanical system, governed by a Hamiltonian $H = T + V$, is said to have an exact (Lie-algebraic) geometrical symmetry g if both the total Hamiltonian, and its kinetic and potential energy parts commute with the operators of g:

$$[H, X_i] = [T, X_i] = [V, X_i] = 0 \ , \ \ g \ni X_i; \ g : \text{Lie-algebra} \ .$$

Similarly: a Hamiltonian H is said to have an algebraic structure g, if H can be expressed in terms of the elements of g.

In this case all the consequences i)–v), as listed in Subsection 2.1.2 are valid. This situation can be characterised in short by saying that the total Hamiltonian, its kinetic and potential energy parts, and its eigenvectors are symmetric.

2.2 Hidden (Dynamical) Symmetry

If a quantum mechanical system is governed by a Hamiltonian H, and it commutes with all the elements of a Lie-algebra:

$$[H, X_i] = 0, \ \ g \ni X_i; \ \ g : \text{Lie-algebra} \ ,$$

but the

$$H = T + V \ , \ [T, X_i] = [V, X_i] = 0$$

equations are not all valid, then the symmetry is called hidden, or dynamical symmetry.

This is still an exact symmetry, i.e. both the Hamiltonian, and its eigenvectors are symmetric. (The operator is scalar, i.e. it commutes with the elements of the algebra, and the eigenvectors transform as the basis vectors of an irreducible representation.)

All the consequences i)-v) of an exact symmetry, as discussed in the previous subsection are valid for this kind of hidden symmetry as well.

Text-book examples of the exact dynamical symmetries are the o(4) symmetry of the Kepler-problem, and the u(3) symmetry of the three-dimensional isotropic harmonic oscillator problem. In this latter case the Hamiltonian is:

$$H = \hbar\omega\Big(N + \frac{3}{2}\Big) \ ,$$

where N is the number (operator) of oscillator quanta (which is an invariant operator of u(3)).

2.3 More Hidden (Broken Dynamical) Symmetry

2.3.1 A Well-Known Example

Let us consider a quantum mechanical system governed by the following Hamiltonian:

$$H = aJ^2 + bJ_0 \ .$$

In this case the three-dimensional rotational symmetry is not valid any more, it is broken by the bJ_0 term, therefore, only the two-dimensional rotations are symmetry-transformations. From the mathematical viewpoint the situation is characterised by the so(2) subgroup of so(3):

$$so(3) \ \supset \ so(2) \ .$$

The solution of the energy-eigenvalue problem is still very simple, the eigenvalues are:

$$E = aJ(J+1) + bM \ ,$$

and the eigenvectors are still the angular-momentum eigenvectors: $|\psi_{JM}\rangle$.

Inspite of the fact that only the so(2) is an exact symmetry, several of the consequencies of the so(3) symmetry survive. The degeneracy of the so(3) multiplets are lifted, but the consequencies of i), ii), and iii) are valid. As for v) it also holds, but it should be put in a little bit more general form: if H has an algebraic structure so(3), and H has a broken dynamical so(3) symmetry, then its eigenvalue-problem has an analytical solution.

In short this situation can be characterised by saying that the operator is not symmetric, but its eigenvectors are still symmetric.

2.3.2 In General

When the Hamiltonian of a quantum mechanical system can be written as a function of the invariant operators of a chain of nested subalgebras:

$$g \supset g_1 \supset g_2 \supset ... \supset g_0 \ ,$$

$$H = f(C_g, C_{g_1}, ..., C_{g_0}) \ ,$$

then we have a special breaking of symmetry ($g, \ g_i, \ i \neq 0$). Then the eigenvectors are more symmetric than the Hamiltonian: only g_0 is an exact symmetry of H, but its eigenvectors carry all the g_i symmetries. We call this situation as a broken dynamical symmetry. In many cases this situation is named simply dynamical symmetry [3–5], nevertheless, it is worth remembering, that it is a more general concept than the text-book definition cited in Subsection 2.2.

Consequences i), ii), and iii) are valid for each g_i symmetry, however, the degeneracy is lifted, except for the degeneracy with respect to the exact symmetry of g_0. The criterium v) for the analytical solution is valid for g in its generalised form: if H has an algebraic structure g, and H has a broken dynamical g symmetry, then its eigenvalue-problem has an analytical solution.

The energy-splitting of the multiplets is in fact a great advantage from the point of view of applications, because it means that non-degenerate spectra can be described by this kind of rather hidden symmetry. Some important examples of the broken dynamical symmetries are:

su^T(2): isospin in nuclear and particle physics,

su(3): in hadronspecroscopy,

su(3): in Elliott's model of light nuclei,

suST(4): spin-isospin symmetry in Wigner's supermultiplets,

u(5): of quadrupole nuclear vibration,

u(6): of the interacting boson model of nuclei,

u(7): three-body-problem, etc.

2.3.3 Elliott's su(3) in Nuclei

In Elliott's model of light nuclei [6] the Hamiltonian contains a harmonic oscillator potential, and the quadrupole-quadrupole interaction of the nucleons, which can be written as a function of the invariant operators of the algebra-chain

$$u(3) \supset su(3) \supset so(3) \ ,$$

as follows [7]:

$$\begin{aligned} H =& H_{OSC} - \theta \sum_{ijm} (-)^m q_{mi} q_{-mj} = \\ =& \hbar\omega(N + \frac{3}{2}) - \theta \sum_{m} (-)^m Q_m Q_{-m} = \\ =& \hbar\omega C^{(1)}_{U3} - 36\theta C^{(2)}_{SU3} + 3\theta C^{(2)}_{SO3} \ . \end{aligned}$$

Here i and j are indices of the nucleons, q is the quadrupole operator of a nucleon, Q is that of the nucleus, $m = -2, -1, 0, 1, 2$, and $C^{(n)}_g$ denotes the Casimir operator of nth degree of the algebra g.

This Hamiltonian splits up the degeneracy of the harmonic oscillator, the energy follows a rotational sequence ($C^{(2)}_{SO3} = J^2$), but states with different su(3) quantum numbers are not mixed.

2.4 Most Hidden (Effective) Symmetry

2.4.1 Motivation

Concerning the usual eigenvalue-problem:

$$H|\psi\rangle = E|\psi\rangle$$

we can put the following question: Is it possible that neither $|\psi\rangle$, nor H is symmetric, yet the symmetry acts?

Surprisingly, the answer is yes. A symmetry can be present and have important physical consequences even in this case [8,9]. This kind of symmetry is called effective symmetry, and it is based on the mathematical concept of embedded representation [10]. The basic idea of the embedded representation can be introduced most easily by using the language of matrix representations.

2.4.2 Embedded Representations: a Simple Example

First we illustrate the concept for the simple example of the so(3) algebra.

We choose a set of basis states defined by the representation indices of the algebras in the chain:

$$so(3) \supset so(2)$$
$$J \ \ , \ \ M \ .$$

The (usual) matrix representation of so(3) is provided by the matrix elements (see Eqs. (3,4)):

$$\langle \psi_{JM} | J_i | \psi_{JM'} \rangle \ = \ f_{MM'}(J)$$

Let us now take a linear combination of the basis states of different so(3) irreps:

$$\phi_M \ = \ \sum_J \ c_J \psi_{JM} \ .$$

If we calculate the matrix elements of the J_i operators between these states, then the result is, in general, not the set of the $f_{M,M'}(J)$ quantities (for any J), which represent the so(3) algebra. In special cases, however, it may happen that the matrix elements give a good approximation to those of the representation matrices (for a value of J). When this is the case, we talk about an (approximate) embedded representation of the so(3) algebra, and related to that we can talk about an (approximate) effective so(3) symmetry.

2.4.3 Embedded Representations: in General

Let α denote the representation index (or indices) of algebra g, and let g' be its subalgebra, with representation index (or indices) β:

$$\begin{array}{ccc} g & \supset & g' \\ \alpha & , & \beta\,. \end{array}$$

(For the sake of simplicity we took a case when the algebra-chain contains only two algebras.)

The (usual) matrix representation of g is provided by the matrix elements:

$$\langle \psi_{\alpha\beta} | \Gamma(X_i^{(g)}) | \psi_{\alpha\beta'} \rangle \; = \; f_{\beta\beta'}(\alpha)\;,$$

where $\Gamma(X_i^{(g)})$'s are operators, which represent the $X_i^{(g)}$ elements of the algebra g. Let us now take a linear combination of the basis states of different irreps of g:

$$\phi_\beta \; = \; \sum_\alpha \, c_\alpha \psi_{\alpha\beta}\;.$$

If we calculate the matrix elements of the $\Gamma(X_i^{(g)})$ operators between these states, then the result is, in general, not the set of the $f_{\beta\beta'}(\alpha)$ quantities, which represent g. In special cases, however, it may happen that the matrix elements give exactly or approximatively the representation matrices. (Please, note that contrary to the simple example of so(3) of the previous subsection, for some other algebras the representation matrices can be obtained exactly.) When this is the case, we talk about an exact or approximate embedded representation of the algebra g, and related to that there is an exact or approximate effective g symmetry.

The linear combination above defines an effective (or average) $\tilde{\alpha}$: representation label (or quantum number).

2.4.4 Consequences

The presence of an effective symmetry has several important consequences:

i) Good effective (average) quantum numbers.

ii) Multiplet structure.

v) If H has an algebraic structure g, and in addition it has an effective symmetry g then the eigenvalue-problem of H has an analytical solution.

The other consequences of the exact symmetries, like the transformation properties of the eigenvectors (iii), and the degeneracy of the multiplets (iv), are not valid, as it is obvious from the introduction of this more general symmetry.

Nevertheless, the existence of effective quantum numbers are of great importance in physical applications. It can explain why some models are succesfull among circumstances, when they have no right to be so. A simple application is mentioned in the next sections.

3 Effective SU(3) for Heavy Nuclei

As it is shown in Ref. [8, 9] an effective su(3) symmetry may survive in heavy nuclei, where the real su(3) symmetry is heavily broken due to the strong spin-orbit and other interactions.

Jarrio et al [9] gave an explicit procedure for the determination of the effective su(3) quantum numbers for well-deformed prolate nuclei. The procedure is based on the application of the asymptotic Nilsson orbits. Hess et al [11] generalised the procedure for oblate nuclei as well as for nuclei with small deformation. The generalization is based on an expansion method in terms of asymptotic Nilsson-orbits.

4 Clusterization and the u(3) Selection Rule

An important question in nuclear cluster (and related decay) studies is whether or not a cluster configuration is present (or allowed) in a specific nuclear state. This question has different aspects, like e.g. the energetics of a cluster configuration, and the compatibility of the microscopic structures (of the nuclear state and the cluster configuration). A considerable insight can be obtained into this latter problem by the applications of simple selection rules.

For the sake of simplicity we address the question if the $C_1 + C_2$ clusterization of (one or another state of) the nucleus P is allowed. The u(3) selection rule works very much in the same way, like the well-known so(3) selection rule.

4.1 so(3) Selection Rule

A nuclear clusterization (or decay) of the type

$$P = C_1 + C_2$$

is allowed if the corresponding angular momenta satisfy the equation:

$$J^P = J^{C_1} \otimes J^{C_2} \otimes J^R ,$$

where J^i denotes the angular momentum of the nucleus i, and J^R stands for that of the relative motion. The $\otimes$ symbols indicate direct product of the representations (vector-additional rule), and '=' should be understood by requiring a matching between J^P and one of the products on the right hand side.

4.2 u(3) Selection Rule for Light Nuclei

The u(3) symmetry is known to be approximatelly good for light nuclei, therefore, an approximate selection rule can be formulated on this basis. The irreducible representations of the u(3) algebra can be labelled by three integer numbers: $[n_1, n_2, n_3]$. Thus the u(3) selection rule for the binary clusterization (decay) reads [12]:

$$[n_1^P, n_2^P, n_3^P] = [n_1^{C_1}, n_2^{C_1}, n_3^{C_1}] \otimes [n_1^{C_2}, n_2^{C_2}, n_3^{C_2}] \otimes [n^R, 0, 0] \ .$$

Since the u(3) symmetry of a nuclear state is determined by its microscopic structure (distribution of the oscillator quanta), this selection rule is based on the microscopic structure, and thus can have non-trivial consequences.

4.3 Simple Applications

The application of the u(3) selection rule is fairly straightforward to the ground-state-like configuration of the light nuclei [12], and they really result in non-trivial conclusions. E.g. the ground state of ^{24}Mg turns out to contain a ^{12}C+^{12}C cluster configuration, but the ^{16}O+^{12}C clusterization is forbidden in the ground state of ^{28}Si, and the ^{16}O+^{16}O configuration is also excluded from the ground state of ^{32}S, in the leading order approximation.

However, these states can not decay by cluster emission due to energetic reasons. Nevertheless, the structural selection rule may have some fingerprints in the experimental observations. In Ref. [13] it is argued that the interference effect of the elastic scattering and elastic transfer of heavy-ions can reflect the allowed or forbidden clusterization of the target nucleus. Further indications to the sensitivity of cluster structure of the ground-state-like configurations of light nuclei come from electrofission and inelastic scattering induced fission studies [14].

In addition to the application to the ground-state-like configurations, the structural selection rule can be usefully applied to other states of nuclei as well. In Ref. [15] for example the alpha-like clusterizations of the super, hyper, etc. deformed states of light nuclei are discussed.

4.4 Deformation-Dependence of Clusterization

In light of the recent experimental interest in the super and hyperdeformed states of light nuclei, a systematic comparision of the allowed clusterization of the normal deformed (ground state), superdeformed and hyperdeformed states of light nuclei can be fruitful. The application of the u(3) selection rule offers a convenient possibility for such studies.

In Ref. [16] the binary clusterizations of the ground-state, superdeformed and hyperdeformed states of ^{40}Ca was investigated. It turns out that in the ground-state the mass-asymmetric cluster-configurations are preferred, inasmuch e.g. ^{28}Si+^{12}C, ^{30}P+^{10}B, ^{32}S+^{8}Be, ^{28}Si+^{12}C, ^{34}Cl+^{6}Li, ^{36}Ar+^{4}He are allowed, while ^{20}Ne+^{20}Ne, ^{22}Na+^{18}F, ^{24}Mg+^{16}O, ^{26}Al+^{14}N are forbidden. For the superdeformed state of this nucleus it is exactly the other way around, the ^{20}Ne+^{20}Ne, ^{22}Na+^{18}F, ^{24}Mg+^{16}O, ^{26}Al+^{14}N configurations are allowed, and the ^{28}Si+^{12}C, ^{30}P+^{10}B, ^{32}S+^{8}Be, ^{28}Si+^{12}C, ^{34}Cl+^{6}Li, ^{36}Ar+^{4}He ones are forbidden. In the hyperdeformed states only the ^{20}Ne+^{20}Ne clusterization is allowed, if we take both clusters in their ground-state.

It is worth noting, that with the application of the u(3) selection one takes into account the deformation of both clusters, as well as that of the parent nucleus, and the nuclear shape can be even triaxial. Furthermore, the different orientation of the clusters are also accounted for. E.g. the ^{20}Ne+^{20}Ne configurations in the superdeformed and hyperdeformed states do not have the same geometrical shape; in the superdeformed state the two clusters with prolate deformation have their symmetry axes perpendicular to the molecular axis, as well as to each other, while in the hyperdeformed state they are lined up: it is a pole-to-pole configuration. In these considerations the Pauli-principle is also fully appreciated.

4.5 Structural Forbiddenness in Binary Fission

Regarding the recent fine-resolution spectroscopic data on the spontaneous fission of heavy nuclei the question of structural forbiddenness is of utmost interest. The simple application of the real u(3) selection rule does not seem to be very convincing, due to the breakdown of this symmetry in heavy nuclei, however, a similar procedure based on the effective u(3) may be promising. It was done for the binary fission modes of the ground-state of ^{252}Cf in Ref. [17]. In addition to the simple yes-and-no answer of the selection rule a quantitative measure was applied for the characterization of the structural forbiddenness, which was defined in such a way, that the information on the cluster-deformation and orientation is not washed out. The result shows that the mass-asymmetric fission modes are definitely preferred by the structural effects.

By applying the effective u(3) selection rule, one can also study the structural effect on the deformation dependence of the mass distribution of spontaneous fission. Preliminary calculations [16] indicate that in the superdeformed state of ^{252}Cf some of the binary cluster configurations become completely allowed, and the mass-distribution shows a double-peaked shape (opposite to the single-peaked form of the ground-state), while for the hyperdeformed state the allowed clusterization shows a broad plateau.

5 Summary

In this contribution we have discussed a way of generalizig the concept of symmetry, starting from the well-known geometrical symmetry, then introducing more and more hidden symmetries, like exact dynamical symmetry, broken dynamical symmetry and effective symmetry. Remarkably, all of these concepts find their application in nuclear struture studies. We mentioned some simple applications based on the selection rules related to these symmetries in the field of nuclear clusterizations. It should be mentioned that up to the broken dynamical symmetries they have extensively been applied [18] in detailed calculations of the spectra, transitions, etc of cluster systems. Similar role for the effective simmetry seems to be possible as well.

A further interesting open question is: to what extent and in which field of physics will this rather general and well-hidden symmetry play a role.

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