Nuclear Theory'22
ed. V. Nikolaev, Heron Press, Sofia, 2003

A q-Deformed Symplectic $sp(4)$ Algebra: How Free Is an Additional Degree of Freedom?

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Abstract.

A q-deformed extension of a fermion realization of the compact symplectic $sp(4)$ algebra is used to describe pairing correlations and higher-order interactions in atomic nuclei. Our results suggest that the q-deformation has physical significance beyond what can be achieved by simply tweaking the parameters of a two-body interaction and the q-parameter. The additional degree of freedom is found to be closely related to isovector pairing correlations between nucleons. The q-deformation also plays a significant role in understanding 'phase transitions' between regions of dominant and negligible higher-order interactions in finite nuclear systems.

1 Introduction

The long-lasting interest in nuclear structure physics is fueled by the fact that the nuclear problem, the many-nucleon non-relativistic Shrödinger equation, cannot be treated exactly. Two major themes help. The analysis of empirical evidence gives rise to simplified or idealized models of physical systems and the recognition of symmetries, exact and approximate, often yield tractable model spaces and exact solutions. Pairing correlations in nuclei possess a clear dynamical symmetry. In the pairing limit the nuclear energy spectra are generated by the conventional $U(2(2j+1)) \supset Sp(2j+1)$ seniority scheme [1,2], or alternatively by the symplectic $Sp(2j+1)$ group together with its dual the symplectic $Sp(4)$ group ($\sim SO(5)$) [3–6]. The latter is an extension to two types of nucleons of Kerman's *quasi-spin* $SU(2)$ group [7] to incorporate proton-neutron pairing correlations.

A recent renaissance of studies on pairing is related to the search of a reliable microscopic theory for a description of medium and heavy nuclei around the $N = Z$ line, where protons and neutrons occupy the same major shells and their mutual interactions are expected to influence significantly the structure and decay of these nuclei. Such a microscopic framework is as well essential for astrophysical applications, for example the description of the rp-process in nucleosynthesis, which runs close to the proton-rich side of the valley of stability through reaction sequences of proton captures and competing β decays [8]. The revival of interest in pairing correlations is also prompted by the initiation of radioactive beam experiments, which advance towards exploration of 'exotic' nuclei, such as neutron-deficient or $N \approx Z$ nuclei far off the valley of stability.

In our search for a microscopic description of pairing in nuclei with mass numbers $32 \leq A \leq 100$ with protons and neutrons filling the same major shell, we employ an $sp(4)$ algebraic model that accounts for proton-neutron and like-particle pairing correlations and higher-J proton-neutron interactions, including the so-called symmetry and Wigner energies [9]. We also extend this model by constructing its q-deformed analog, the $sp_q(4)$ algebraic model [10, 11]. Since the dawn of the q-deformed (quantum) algebra concept [12] the recognition of two major features makes the approach very attractive for physical applications. The first is that in the $q \rightarrow 1$ limit of the deformation parameter the q-algebra reverts back to the "classical" Lie algebra. Second, the q-deformation introduces richer structures into the theory while preserving the underlying symmetry. We consider a q-deformation of the building blocks of a two-body Hamiltonian without compromising fundamental symmetries inherent to the quantum mechanical theory. In such scenario the q-deformation accounts for non-linear contributions of higher-order (many-body) interactions without affecting physical observables.

In what follows, the q-deformed algebraic model and its "classical" limit are reviewed briefly. Next, we present an overview of the major applications of the $Sp(4)$ model as well as novel properties of q-deformation as introduced by the non-linear $sp_q(4)$ algebraic approach.

2 The algebraic $sp_{(q)}(4)$ pairing model

The $sp_q(4)$ algebra [10, 11] (which reverts to the "classical" $sp(4)$ Lie algebra in the $q \rightarrow 1$ limit) is realized in terms of creation/annihilation fermion operators ($\alpha^{(\dagger)}_{jm\sigma} \rightarrow c^{(\dagger)}_{jm\sigma}$) with a non-zero anticommutation relation

$$\{\alpha_{j',m,\sigma}, \alpha^{\dagger}_{j,m',\sigma}\}_{q^{\pm 1}} = q^{\pm \frac{N_\sigma}{2\Omega}} \delta_{j,j'} \delta_{m,m'} \ , \qquad (1)$$

where $\sigma = \pm 1/2$ distinguishes between protons and neutrons, j is total angular momentum (half-integer) with a third projection m, and $2\Omega = \Sigma_j(2j+1)$ is the shell dimension. The operator, $N_{\pm 1} = \sum_{m=-j}^{j} c^{\dagger}_{jm,\pm 1/2} c_{jm,\pm 1/2}$, that enters in (1), counts the total number of protons (neutrons).

In addition to the number operator, $N = N_1 + N_{-1}$, and the third projection of isospin, $T_0 = (N_1 - N_{-1})/2$, the basis operators in $sp_q(4)$ are

$$T_{\pm} = \frac{1}{\sqrt{2\Omega}} \sum_{jm} \alpha^{\dagger}_{jm,\pm 1/2} \alpha_{jm,\mp 1/2} \quad (2)$$

$$A^{\dagger}_{\mu=\sigma+\sigma'} = \frac{1}{\sqrt{2\Omega(1+\delta_{\sigma\sigma'})}} \sum_{jm} (-1)^{j-m} \alpha^{\dagger}_{jm,\sigma} \alpha^{\dagger}_{j,-m,\sigma'},\ A_{\mu} = (A^{\dagger}_{\mu})^{\dagger}\ . \quad (3)$$

The operators $T_{0,\pm}$ are related to isospin, while $A^{\dagger}$ (A) create (annihilate) a pair of total angular momentum $J^{\pi} = 0^{+}$ and isospin T=1. A model Hamiltonian with an $Sp_q(4)$ dynamical symmetry, that preserves number of particles, third projection of isospin and angular momentum,

$$H_q = -\varepsilon^q N - G A^{\dagger}_0 A_0 - F(A^{\dagger}_{+1} A_{+1} + A^{\dagger}_{-1} A_{-1}) - \frac{1}{2} E \Big(\frac{T^2}{\Omega} - \left[\frac{N}{2\Omega} \right] \Big)$$
$$- C 2\Omega \left[\frac{1}{\Omega} \right] \Big(\left[\frac{N}{2} - \Omega \right]^2_{\frac{1}{2\Omega}} - [\Omega]^2_{\frac{1}{2\Omega}} \Big) - \Big(D - \frac{E_q}{2\Omega} \Big) \Omega \left[\frac{1}{\Omega} \right] [T_0]^2_{\frac{1}{2\Omega}} \quad (4)$$

$$H_q \stackrel{q \to 1}{\to} H_{cl}\ , \quad (5)$$

includes a two-body isovector (T=1) pairing interaction and a diagonal isoscalar (T=0) force, which is proportional to a symmetry and Wigner term ($T(T$+1)-like dependence), as well as small isospin violating nuclear interactions, in addition to higher-order many-body interactions prescribed by q-deformation [9]. By definition, $[X]_k = \frac{q^{kX} - q^{-kX}}{q^k - q^{-k}}$. The interaction strength parameters in (4) coincide with those for (5) and have a smooth dependence on the mass A,

$$\frac{G}{\Omega} = \frac{25.7 \pm 0.5}{A}, \quad \frac{F}{\Omega} = \frac{23.9 \pm 1.1}{A}, \quad \frac{E}{2\Omega} = \frac{-52 \pm 5}{A},$$
$$D = \frac{-37 \pm 5}{A} + (-0.24 \pm 0.09), \quad C = \left(\frac{32 \pm 1}{A} \right)^{1.7 \pm 0.2}. \quad (6)$$

The basis states are constructed as (T=1)-paired fermions,

$$|n_1, n_0, n_{-1})_q = \left(A^{\dagger}_1 \right)^{n_1} \left(A^{\dagger}_0 \right)^{n_0} \left(A^{\dagger}_{-1} \right)^{n_{-1}} |0\rangle\ , \quad (7)$$

and model the 0^{+} ground state for even-even and some odd-odd nuclei and the corresponding isobaric analog excited 0^{+} state for even-A nuclei. We refer to these states as *isovector-paired* 0^{+} states [9].

3 The "Classical" Limit: Applications of The $Sp(4)$ Model

3.1 Classification Scheme

The non-deformed algebraic model provides for a natural and convenient classifications scheme of nuclei and their states. The classification allows for a large-scale systematic study of nuclear properties and phenomena observed.

In general, each (μ=$T,0,\pm$) realization of the reduction chain of the $sp(4)$ algebra, $sp(4)\supset u^{\mu}(2)\supset su^{\mu}(2)$, describes a limiting case of a restricted symmetry: isospin symmetry, proton-neutron (pn) pairing and like-particle pairing, respectively. It provides for a complete labeling of the basis vectors of a (w=Ω,t=0) irreducible representations[1] of $Sp(4)$ by the eigenvalues of the invariant operators of the underlying subalgebras. The first-order $C_1^{\{\mu=T,0,\pm\}}$=$\{N,T_0,N_{\mp 1}\}$ invariant of $u^{\mu}(2)$ reduces the finite action space into a direct sum of unitary irreps of $U^{\mu}(2)$ labeled by the C_1^{μ} eigenvalue ($\{n,i,N_{\mp}\}$-multiplets). The next two quantum numbers are provided by the $SU^{\mu}(2)$ group in a standard way: the "spin", s (related to the eigenvalue of the second-order Casimir invariant of $su^{\mu}(2)$) and its third projection.

Table 1. Classification scheme of even-A nuclei and their *isovector paired* states in the $1f_{7/2}$ shell, $\Omega_{7/2}$=4. The shape of the table is symmetric with respect to the sign of i and $n-2\Omega$. The basis states are labeled by the numbers of particle pairs $|n_1,n_0,n_{-1})$ (7). The subsequent action of the $SU^{\mu}(2)$ generators (shown in parenthesis) constructs the constituents in a given $SU^{\mu}(2)$ multiplet (μ=$T,0,\pm$).

n		$i=0$	$i=-1$	$i=-2$	$i=-3$	$i=-4$
0		$\|0,0,0)$ $^{40}_{20}\mathrm{Ca}_{20}$				
2	↙	$\|0,1,0)$ $^{42}_{21}\mathrm{Sc}_{21}$	$\|0,0,1)$ $^{42}_{20}\mathrm{Ca}_{22}$			
4	...	$\|1,0,1)$ $\|0,2,0)$ $^{44}_{22}\mathrm{Ti}_{22}$	$\|0,1,1)$ $^{44}_{21}\mathrm{Sc}_{23}$	$\|0,0,2)$ $^{44}_{20}\mathrm{Ca}_{24}$		
6	...	$\|1,1,1)$ $\|0,3,0)$ $^{46}_{23}\mathrm{V}_{23}$	$\|1,0,2)$ $\|0,2,1)$ $^{46}_{22}\mathrm{Ti}_{24}$	$\|0,1,2)$ $^{46}_{21}\mathrm{Sc}_{25}$	$\|0,0,3)$ $^{46}_{20}\mathrm{Ca}_{26}$	
8	$\leftarrow^{(T_+)}$	$\|2,0,2)$ $\|1,2,1)$ $\|0,4,0)$ $^{48}_{24}\mathrm{Cr}_{24}$	$\|1,1,2)$ $\|0,3,1)$ $^{48}_{23}\mathrm{V}_{25}$	$\|0,2,2)$ $\|1,0,3)$ $^{48}_{22}\mathrm{Ti}_{26}$	$\|0,1,3)$ $^{48}_{21}\mathrm{Sc}_{27}$	$\|0,0,4)$ $^{48}_{20}\mathrm{Ca}_{28}$
	$(A^{\dagger}_{-1})$ ↘	↓ $(A^{\dagger}_{0})$	⋮	⋮	↙ $(A^{\dagger}_{+1})$	

[1]The two quantum numbers, w and t (t is the Flower's reduced isospin), label an irreducible representation of $Sp(4)$. The irrep with t=0 corresponds to the fully-paired case (7).

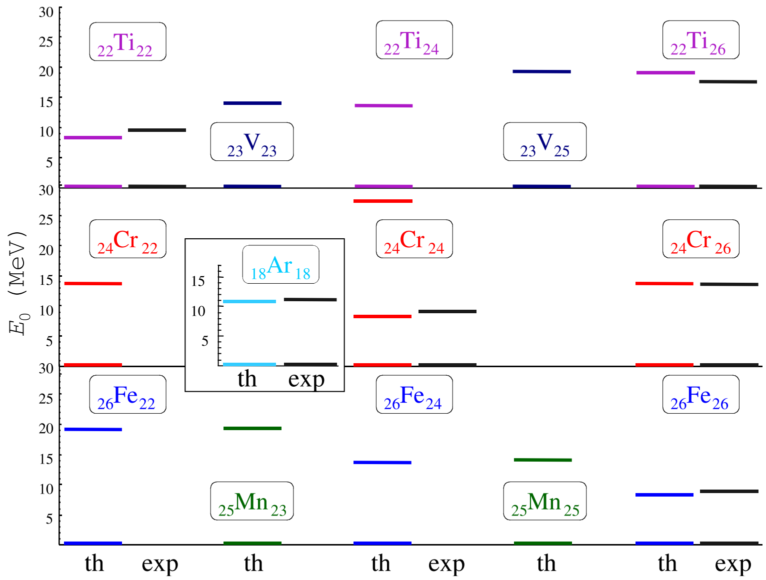

Figure 1. Theoretical ('th') and experimental ('exp') energy spectra of the higher-lying *isovector-paired* 0^+ states for isotopes in $1f_{7/2}$ (in $1d_{3/2}$ (insert)).

A vector with fixed quantum numbers, (N_+, N_-) or (n, i), corresponds to a given nucleus (a cell in Table 1). In this way the $Sp(4)\supset U(2)\supset SU(2)$ symmetry provides for a natural classification scheme of nuclei that belong to a single-j level, which are mapped to algebraic $U(2)$ multiplets. This classification also extends to the corresponding ground and excited states of the nuclei, which can be distinguished as eigenstates of the Casimir operators of $su(2)$ in the limiting cases.

3.2 Energy Spectra of *Isovector-Paired* 0^+ States

The $Sp(4)$ model leads to a very good reproduction of the experimental energies of the lowest *isovector-paired* 0^+ state for even-A nuclei with nuclear masses $32\leq A \leq 164$ [9]. Without varying the values of the interaction strength parameters, the energy of the higher-lying *isovector-paired* 0^+ states can be theoretically calculated and they agree remarkably well with the available experimental values for the single-j $1d_{3/2}$ and $1f_{7/2}$ orbits[1] (Figure 1). This agreement, which is observed not only in single cases but throughout the shells, represents an astonishing result. Since the higher-lying *isovector paired* 0^+ states constitute an experimental set independent of the data that determines the interaction strength

[1]The energy spectra of nuclei in the region with nuclear masses $56<A<164$ is not yet completely measured, especially the higher-lying 0^+ states.

parameters in (5), such an result is, first, an independent test of the physical validity of the strength parameters, and, second, an indication that the interactions interpreted by the model Hamiltonian are the main driving force that defines the properties of these states. In this way, the simple $Sp(4)$ model provides for a reasonable prediction of the *isovector paired* (ground and/or excited) 0^+ states in proton-rich nuclei with energy spectra not yet experimentally fully explored.

3.3 Fine Effects: $N = Z$ Irregularities, Staggering Behavior and Pairing Gap

The theoretical $Sp(4)$ model is further tested through second- and higher-order discrete derivatives of the energies of the lowest *isovector-paired* 0^+ states in the $Sp(4)$ systematics, without any parameter variation. The proposed model is used to successfully interpret: the two-proton (two-neutron) separation energy $S_{2p(2n)}$ for even-even nuclei (hence determined the two-proton drip line), the S_{pn} energy difference when a pn $T = 1$ pair is added, the observed irregularities around N=Z [13], the prominent "ee-oo" staggering between even-even and odd-odd nuclides, the like-particle and pn isovector pairing gaps, and the large contribution to the finite energy differences due to J=1 and higher-J pn interactions [14]. We suggest that the oscillating "ee-oo" effects correlate with the alternating of the seniority numbers related to the pn and like-particle isovector pairing, which is in addition to the larger contribution due to the discontinuous change in isospin values associated with the symmetry energy.

The theoretical discrete derivatives under investigation not only follow the

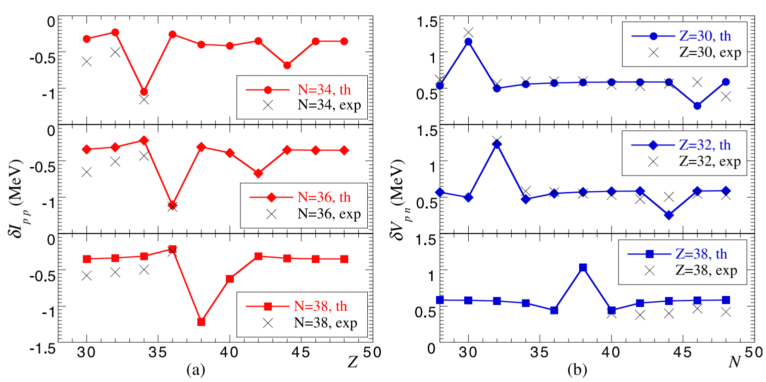

Figure 2. Second discrete derivatives of energy: (a) $-\delta I_{pp(nn)}(N_\pm) = \left[E_0(N_\pm + 2) - 2E_0(N_\pm) + E_0(N_\pm - 2)\right]/4$ versus $N_\pm$, as an estimate for the non-pairing like-particle nuclear interaction in MeV for the $N(Z)$=34,36,38–multiplets; (b) $-\delta V_{pn}(N_+, N_-)$=$\left[E_0(N_+ + 2, N_- + 2) - E_0(N_+ + 2, N_-) - E_0(N_+, N_- + 2) + E_0(N_+, N_-)\right]/4$ versus N_+ and N_-, as an estimate for the residual interaction between the last proton and the last neutron in MeV for Zn, Ge, Sr isotopes.

experimental patterns but their magnitude is found to be in a remarkable agreement with the data. The present study brings forward a very useful result. We find a finite energy difference,

$$E_0(n,i+1) - 2E_0(n,i) + E_0(n,i-1) = \\ = E_0(N_+ + 1, N_- - 1) - 2E_0(N_+, N_-) + E_0(N_+ - 1, N_- + 1)\ , \quad (8)$$

that, for the specific case $i = 0$ (or N=Z), can be interpreted as an isovector pairing gap, $\tilde{\Delta}$=Δ_{pp}+Δ_{nn}–$2\Delta_{pn}$, which is related to the like-particle and pn isovector pairing gaps. Indeed, they correspond to the T=1 pairing mode because we do not consider the binding energies for all the nuclei but the respective isobaric analog 0^+ states for the odd-odd nuclei with a $J{\neq}0^+$ ground state. This investigation is the first of its kind. Moreover, the relevant energies are corrected for the Coulomb interaction and therefore the isolated effects reflect solely the nature of the nuclear interaction. Small deviations from the experimental data are attributed to other two-body interactions or higher-order correlations that are not included in the theoretical model.

3.4 Isospin Breaking and Forbidden β Decays

Laboratory studies of non-analog Fermi (ΔJ=0) β-decay transitions $0^+ \rightarrow 0^+$ provide for an excellent test of the isospin admixture. If there was no isospin mixing, any β transition between nuclear states belonging to different T multiplets would be forbidden. The experimental results clearly reveal the existence of isospin mixing [15]. A very interesting result follows from the estimate of the nuclear Fermi β decay rate. When compared to the decay rate for purely leptonic muon decay, it determines a value of the Cabbibo-Kobayashi-Maskawa (CKM) mixing matrix element between u and d quarks (v_{ud}). This in turn furnishes a precise test of the unitary condition of the CKM matrix under the assumption of the three-generation standard particle model.

The small mixing of the 0^+ isospin eigenstates of the model $Sp(4)$ Hamiltonian (5) from different isospin multiplets yields very small but non-zero transition matrix elements for non-analog $\beta^\pm$ decay transitions between the *isovector paired* 0^+ states. For the nuclei in the $1f_{7/2}$ shell, such transitions to the 0^+ ground state of the daughter nucleus (or the lowest isobaric analog 0^+ state for ^{48}Mn and ^{48}V) are suppressed but not improbable. Although the detection of the ΔT=2 non-analog transitions is hindered by the higher isospin-mixing governed by the Coulomb interaction, the theoretical $Sp(4)$ model suggests the existence of β-decay branches to these non-analog states.

4 Novel Features of q-Deformation

4.1 Decoupling of the q-Deformed Parameter

Varying the q-deformation parameter affects the interaction strengths very little and hence q and the interaction strength parameters are decoupled. This means that, in the "classical" picture (5), the two-body nuclear interaction strengths, can be assigned the best-fit global values for the model space under consideration without compromising overall quality of the theory. The same values, due to the decoupling, are assigned to the strength parameters in the q-deformed model (4). In this way, the corresponding q-deformed Hamiltonian possesses a precious asset, namely it contains in itself exactly the two-body "classical" Hamiltonian (5). While leaving the strength of the two-body interactions unchanged, the q-deformation allows one to take into account, in a prescribed way, complicated higher-order many-body interactions. In a word, this observation underscores the fact that the deformation represents something fundamentally different, a feature that cannot be "mocked up" by allowing the strengths of the non-deformed interaction to absorb its effect.

4.2 Physical Significance of q-Deformation

Higher-order interactions in nuclei can be investigated via the use of a local q-parameter that is allowed to vary within each individual nucleus. In comparison to experiment , the solutions for the deformation parameter $|\varkappa|$ are found to fall on a smooth curve that tracks with the energy of the lowest 2_1^+ states [16]. This outcome is significant in two aspects. First, the smooth behavior of the q-parameter with respect to a change in the proton (neutron) number suggests that the nature of the q-deformation is deeply rooted in the basic internucleon interactions and is not sporadic in character. The second aspect is related to the lowest 2_1^+ states. These energies are largest near closed shells where the pairing effect is essential for determining the low-lying spectrum and decrease with increasing collectivity and shape deformation. Similar behavior is suggested for the q-deformation in the sense that dominant pairing correlations are accompanied by non-negligible many-body interactions as prescribed by H_q (4), while long-range collectivity suppresses their overall contribution. In a word, the observed smooth behavior of the q-parameter, even though a qualitative result, gives some insight into the understanding of the nature of the q-deformation and reveals its functional dependence on the model quantum numbers.

The analysis yields that the many-body nature of the interaction is most important around closed shells and the regions with $N_+ \approx N_-$. For these nuclei the q-parameter has significant values and the experimental energies can be reproduced exactly. An interesting point is that q tends to peak for even-even nuclei when $N_+ = N_-$ where strong pairing correlations are expected. The behavior of the q-parameter is persistent in both regions under consideration, namely $1f_{7/2}$

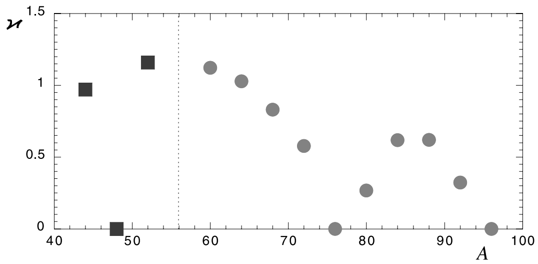

Figure 3. The $\varkappa$ deformation parameter as a function of the nuclear mass A for the even-even $N = Z$ nuclei in the $1f_{7/2}$ level (boxes) and $1f_{5/2}2p_{1/2}2p_{3/2}1g_{9/2}$ major shell (dots) (separated by a dotted line).

and $1f_{5/2}2p_{1/2}2p_{3/2}1g_{9/2}$ shells, which is best seen for the $N = Z$ nuclei (Figure 3) as the i=0 multiplet evolves continuously from the first orbit to the next shell (with ^{56}Ni being the last nucleus in the multiplet in $1f_{7/2}$ and the first for the $1f_{5/2}2p_{1/2}2p_{3/2}1g_{9/2}$ major shell).

"Classical" values of the q-deformation parameter ($q\approx 1$) are found in nuclei with only one or two particle/hole pairs from a closed shell. This is an expected result since the number of particles is insufficient to sample the effect of higher-order terms in a q-deformed interaction. For these nuclei the non-deformed limit gives a good description. Around mid-shell ($N\approx 2\Omega$) the q-deformation adds little improvement to the theory with the experimental values remaining close to the "classical" limit. This suggests that for these nuclei the many-body interactions as prescribed by the q-deformation are negligible. The results imply that even though the q-parameter gives additional freedom for all the nuclei, it only improves the model around regions of dominant pairing correlations.

In short, the results suggest that the deformation has physical significance over-and-above the simple pairing gap concept, extending to the very nature of the nuclear interaction itself and beyond what can be achieved by simply tweaking the parameters of a two-body interaction. The specific non-linear features of the nuclear structure can be investigated through the use of a *local* q value, which is in addition to the good description of the *global* properties of the nuclear dynamics provided by the "classical" two-body interaction.

4.3 q-Deformed Parameter and 'Phase Transition'

As described above, higher-order interactions in nuclei such as the ones investigated via the use of the local q-parameter are found to develop smoothly with mass number and be nucleus-dependent in their nature. Throughout a major shell for even-even nuclei, they outline two regions, one of negligible higher-order correlations (I) and another where the latter are significant (II). While in the first 'phase' (I) the deformation parameter $\varkappa$ is zero, it is considerable in (II) where non-linear many-body fields are required. In analogy with statistical mechanics, the q-parameter appears as an order parameter, $\varkappa$ (q=$e^{\varkappa}$), for a 'phase transition' between the 'phases' (I) and (II). A criticism may be raised here as we talk about phase transitions in small finite systems such as nuclei. Small systems do not exhibit true phase transitions (as defined within the framework of statistical mechanics). However, it is reasonable to say that they can have an order parameter: such that represents the symmetry (order) in a quantum mechanical system and reflects any changes in the properties of the system, which are governed by this symmetry.

Zero values of $\varkappa$ are observed for comparatively higher E_{4^+}/E_{2^+} ratios and strong non-linear effects are found for lower E_{4^+}/E_{2^+} ratios, where the E_{4^+}/E_{2^+} quantity is indicative of the collectivity of the system ($E_{4^+}/E_{2^+}\approx 1$ for nuclei in the seniority regime, $E_{4^+}/E_{2^+}\approx 2$ for (anharmonic) vibrational nuclei and $E_{4^+}/E_{2^+}\approx$3.33 for well-deformed rotational nuclei [17]). Typically, $\varkappa$ tends to zero around $E_{4^+}/E_{2^+}\approx$2-2.5, where the 'phase transition' occurs. Hence, the behavior of the order parameter with changing E_{4^+}/E_{2^+} ratio shows an abrupt zeroing of $\varkappa$ soon after collectivity develops.

4.4 Smooth Dependence of q on Nuclear Characteristics

q-Deformation, and hence the development of non-linear effects, is related in a non-trivial way to the underlying nuclear structure. In general, the many-body interactions yield very complicated matrix elements and the analytical modeling of some of them is made possible due to the quantum extension of $sp(4)$. For even-even nuclei a functional dependence of $\varkappa$ on the total number of particles n and the isospin projection i is found in the form

$$\varkappa(n,i) = \mathcal{A}(\frac{n}{2\Omega}-1)(\frac{n}{2\Omega}+\mathcal{B}-2\Theta(n-2\Omega))e^{-0.5\left(\frac{i}{\mathcal{C}/2}\right)^2} + \mathcal{D}\Theta(n-2\Omega)|i|\sqrt{\frac{n}{2\Omega}-1}\,, \quad (9)$$

where $\Theta(x)$ is the step-function defined as $\Theta(x)$=1(0) when $x\geq 0$ ($x<0$). A fit (χ=0.13) to the values of $\varkappa$ for the even-even nuclei in the major shell, $1f_{5/2}2p_{1/2}2p_{3/2}1g_{9/2}$, estimates the parameters of the analytical function to be

$$\mathcal{A}=-2.86, \quad \mathcal{B}=0.21, \quad \mathcal{C}=2.46, \quad \mathcal{D}=0.12. \quad (10)$$

The dependence of $\varkappa$ on n and i (9) reveals that the non-linear effects contribute significantly to the energy peak at Z=N along with two-body pn correlations.

In conclusion, the symplectic $Sp(4)$ scheme allows not only for an extensive systematic study of various experimental patterns of the even-A nuclei, it also offers simple $sp(4)$ and $sp_q(4)$ algebraic models for interpreting the results. The outcome of the present investigation shows that, in comparison to experiment, the $sp(4)$ algebraic approach reproduces not only global trends of the relevant energies but as well the smaller fine features driven by isovector pairing correlations and higher-J pn and like-particle nuclear interactions. In addition to this, the variations within individual nuclei due to higher-order many-body interactions are described by the non-linear q-deformed $sp_q(4)$ theory with a local q-parameter fundamentally linked to the very nature of the nuclear interaction.

Acknowledgments

This work was supported by the US National Science Foundation through Grant No. 0140300. KDS acknowledges supplemental support from the Graduate School of Louisiana State University.

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