The X(3) Model and its Connection to the Shape/Phase Transition Region of the Interacting Boson Model

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Abstract. A γ-rigid version (with $\gamma = 0$) of the X(5) critical point symmetry is constructed. The model, to be called X(3) since it is proved to contain three degrees of freedom, utilizes an infinite well potential, is based on exact separation of variables, and leads to parameter free (up to overall scale factors) predictions for spectra and $B(E2)$ transition rates, which are in good agreement with existing experimental data for ^{172}Os and ^{186}Pt. The predictions of X(3) are furthermore compared to two-parameter calculations in the framework of the Interacting Boson Approximation (IBA) model. The results show that X(3) coincides with IBA parameters consistent with the phase/shape transition region of the IBA. The same turns out to hold also for the parameter independent (up to overall scale factors) predictions of the X(5)-β^2 and X(5)-β^4 models, which are variants of the X(5) critical point symmetry developed within the framework of the geometric collective model, manifested experimentally in ^{146}Ce, ^{174}Os, and ^{158}Er, ^{176}Os respectively.

1 Introduction

Critical point symmetries [1, 2], describing nuclei at points of shape/phase transitions between different limiting symmetries, have recently attracted considerable attention, since they lead to parameter independent (up to overall scale factors) predictions which are found to be in good agreement with experiment [3–6]. The X(5) critical point symmetry [2], in particular, is supposed to correspond to the transition from vibrational [U(5)] to prolate axially symmetric [SU(3)] nuclei, materialized in the $N = 90$ isotones ^{150}Nd, ^{152}Sm, ^{154}Gd, and ^{156}Dy.

On the other hand, it is known that in the framework of the nuclear collective model [7], which involves the collective variables β and γ, interesting special cases occur by “freezing” the γ variable [8] to a constant value.

In the present work we construct a version of the X(5) model in which the γ variable is "frozen" to $\gamma = 0$, instead of varying around the $\gamma = 0$ value within a harmonic oscillator potential, as in the X(5) case. It turns out that only three variables are involved in the present model, which is therefore called X(3). Exact separation of the β variable from the angles is possible. Experimental realizations of X(3) appear to occur in ^{172}Os and ^{186}Pt. The results are also compared to Interacting Boson Model (IBM) [11] two-parameter calculations, showing that they are consistent with IBM parameters close to the phase/shape transition region of the IBM.

2 The X(3) Model

In the collective model of Bohr [7] the classical expression of the kinetic energy corresponding to β and γ vibrations of the nuclear surface plus rotation of the nucleus has the form [7, 9]

$$T = \frac{1}{2}\sum_{k=1}^{3} \mathcal{J}_k\, \omega_k'^2 + \frac{B}{2}\,(\dot\beta^2 + \beta^2\dot\gamma^2), \tag{1}$$

where β and γ are the usual collective variables, B is the mass parameter,

$$\mathcal{J}_k = 4B\beta^2 \sin^2\!\left(\gamma - \tfrac{2}{3}\pi k\right) \tag{2}$$

are the three principal irrotational moments of inertia, and ω_k' ($k = 1, 2, 3$) are the components of the angular velocity on the body-fixed k-axes, which can be expressed in terms of the time derivatives of the Euler angles $\dot\phi, \dot\theta, \dot\psi$ [9]

$$\begin{aligned}
\omega_1' &= -\sin\theta\cos\psi\,\dot\phi + \sin\psi\,\dot\theta,\\
\omega_2' &= \sin\theta\sin\psi\,\dot\phi + \cos\psi\,\dot\theta,\\
\omega_3' &= \cos\theta\,\dot\phi + \dot\psi.
\end{aligned} \tag{3}$$

Assuming the nucleus to be γ-rigid (i.e. $\dot\gamma = 0$), as in the Davydov and Chaban approach [8], and considering in particular the axially symmetric prolate case of $\gamma = 0$, we see that the third irrotational moment of inertia $\mathcal{J}_3$ vanishes, while the other two become equal $\mathcal{J}_1 = \mathcal{J}_2 = 3B\beta^2$, the kinetic energy of Eq. (1) reaching the form [9, 10]

$$T = \frac{1}{2}3B\beta^2(\omega_1'^2 + \omega_2'^2) + \frac{B}{2}\,\dot\beta^2 = \frac{B}{2}\Big[3\beta^2(\sin^2\theta\,\dot\phi^2 + \dot\theta^2) + \dot\beta^2\Big]. \tag{4}$$

It is clear that in this case the motion is characterized by three degrees of freedom. Introducing the generalized coordinates $q_1 = \phi$, $q_2 = \theta$, and $q_3 = \beta$, the kinetic energy becomes a quadratic form of the time derivatives of the generalized coordinates [9, 12]

$$T = \frac{B}{2}\sum_{i,j=1}^{3} g_{ij}\ \dot q_i \dot q_j, \tag{5}$$

with the matrix g_{ij} having a diagonal form

$$g_{ij} = \begin{pmatrix} 3\beta^2 \sin^2\theta & 0 & 0 \\ 0 & 3\beta^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

(In the case of the full Bohr Hamiltonian [7] the square matrix g_{ij} is 5-dimensional and non-diagonal [9, 12].) Following the general procedure of quantization in curvilinear coordinates one obtains the Hamiltonian operator [9, 12]

$$H = -\frac{\hbar^2}{2B}\,\Delta + U(\beta) = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^2}\frac{\partial}{\partial\beta}\beta^2\frac{\partial}{\partial\beta} + \frac{1}{3\beta^2}\Delta_\Omega\right] + U(\beta), \tag{7}$$

where Δ_Ω is the angular part of the Laplace operator

$$\Delta_\Omega = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{8}$$

The Schrödinger equation can be solved by the factorization

$$\Psi(\beta,\theta,\phi) = F(\beta)\,Y_{LM}(\theta,\phi), \tag{9}$$

where $Y_{LM}(\theta,\phi)$ are the spherical harmonics. Then the angular part leads to the equation

$$-\Delta_\Omega Y_{LM}(\theta,\phi) = L(L+1)Y_{LM}(\theta,\phi), \tag{10}$$

where L is the angular momentum quantum number, while for the radial part $F(\beta)$ one obtains

$$\left[\frac{1}{\beta^2}\frac{d}{d\beta}\beta^2\frac{d}{d\beta} - \frac{L(L+1)}{3\beta^2} + \frac{2B}{\hbar^2}\Big(E - U(\beta)\Big)\right]F(\beta) = 0. \tag{11}$$

As in the case of X(5) [2], the potential in β is taken to be an infinite square well

$$U(\beta) = \begin{cases} 0, & 0 \le \beta \le \beta_W \\ \infty, & \beta > \beta_W \end{cases}, \tag{12}$$

where β_W is the width of the well. In this case $F(\beta)$ is a solution of the equation

$$\left[\frac{d^2}{d\beta^2} + \frac{2}{\beta}\frac{d}{d\beta} + \left(k^2 - \frac{L(L+1)}{3\beta^2}\right)\right]F(\beta) = 0 \tag{13}$$

in the interval $0 \le \beta \le \beta_W$, where reduced energies $\varepsilon = k^2 = 2BE/\hbar^2$ [2] have been introduced, while it vanishes outside. Substituting $F(\beta) = \beta^{-1/2}f(\beta)$ one obtains the Bessel equation

$$\left[\frac{d^2}{d\beta^2} + \frac{1}{\beta}\frac{d}{d\beta} + \left(k^2 - \frac{\nu^2}{\beta^2}\right)\right]f(\beta) = 0, \tag{14}$$

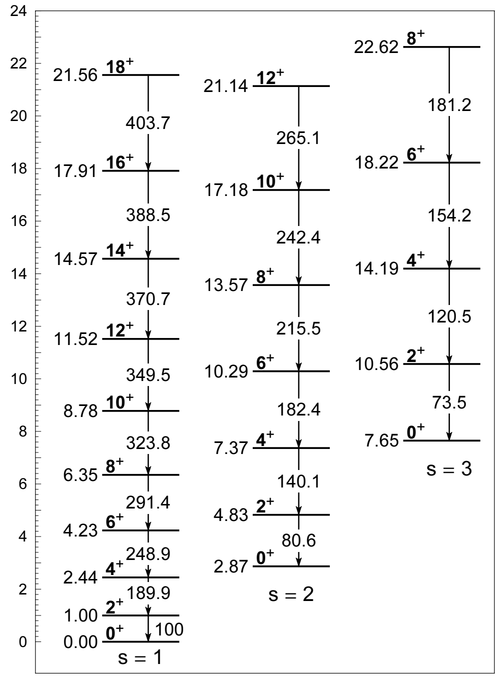

Figure **1.** Energy levels of the ground state ($s = 1$), β_1 ($s = 2$), and β_2 ($s = 3$) bands of X(3), normalized to the energy of the lowest excited state, 2_1^+, together with intraband $B(E2)$ transition rates, normalized to the transition between the two lowest states, $B(E2; 2_1^+ \to 0_1^+)$.

where

$$\nu = \sqrt{\frac{L(L+1)}{3} + \frac{1}{4}}, \tag{15}$$

the boundary condition being $f(\beta_W) = 0$. The solution of (13), which is finite at $\beta = 0$, is then

$$F(\beta) = F_{sL}(\beta) = \frac{1}{\sqrt{c}}\,\beta^{-1/2} J_\nu(k_{s,\nu}\beta), \tag{16}$$

with $k_{s,\nu} = x_{s,\nu}/\beta_W$ and $\varepsilon_{s,\nu} = k_{s,\nu}^2$, where $x_{s,\nu}$ is the s-th zero of the Bessel function of the first kind $J_\nu(k_{s,\nu}\beta_W)$ and the normalization constant $c = \beta_W^2\, J_{\nu+1}^2(x_{s,\nu})/2$ is obtained from the condition $\int_0^{\beta_W} F_{sL}^2(\beta)\,\beta^2 d\beta = 1$. The corresponding spectrum, shown in Figure 1, is then

$$E_{s,L} = \frac{\hbar^2}{2B}\,k_{s,\nu}^2 = \frac{\hbar^2}{2B\beta_W^2}\,x_{s,\nu}^2. \tag{17}$$

It should be noticed that in the X(5) case [2] the same Eq. (14) occurs, but with $\nu = \sqrt{\frac{L(L+1)}{3} + \frac{9}{4}}$.

From the symmetry of the wave function of Eq. (9) with respect to the plane which is orthogonal to the symmetry axis of the nucleus and goes through its center, follows that the angular momentum L can take only even nonnegative values. Therefore no γ-bands appear in the model, as expected, since the γ degree of freedom has been frozen.

In the general case the quadrupole operator is

$$T_\mu^{(E2)} = t\,\beta\Big[D^{2\,*}_{\mu,0}(\Omega)\cos\gamma + \frac{1}{\sqrt{2}}[D^{2\,*}_{\mu,2}(\Omega) + D^{2\,*}_{\mu,-2}(\Omega)]\sin\gamma\Big], \tag{18}$$

where Ω denotes the Euler angles and t is a scale factor. For $\gamma = 0$ the quadrupole operator becomes

$$T_\mu^{(E2)} = t\,\beta\sqrt{\frac{4\pi}{5}}\,Y_{2\mu}(\theta,\phi). \tag{19}$$

$B(E2)$ transition rates

$$B(E2; sL \to s'L') = \frac{1}{2L+1}\left|\langle s'L'||T^{(E2)}||sL\rangle\right|^2 \tag{20}$$

are calculated using the wave functions of Eq. (9) and the volume element $d\tau = \beta^2\sin\theta\,d\beta d\theta d\phi$, the final result being

$$B(E2; sL \to s'L') = t^2\left(C^{L'\,0}_{L\,0,\,2\,0}\right)^2 I^2_{sL;\,s'L'}, \tag{21}$$

where $C^{L'\,0}_{L\,0,\,2\,0}$ are Clebsch–Gordan coefficients and the integrals over β are

$$I_{sL;\,s'L'} = \int_0^{\beta_W} \beta\,F_{sL}(\beta)\,F_{s'L'}(\beta)\,\beta^2\,d\beta. \tag{22}$$

3 The IBA Hamiltonian and Symmetry Triangle

The study of shape/phase transitions in the IBA is facilitated by writing the IBA Hamiltonian in the form [13, 14]

$$H(\zeta,\chi) = c\left[(1-\zeta)\hat{n}_d - \frac{\zeta}{4N_B}\hat{Q}^\chi\cdot\hat{Q}^\chi\right], \tag{23}$$

where $\hat{n}_d = d^\dagger\cdot\tilde{d}$, $\hat{Q}^\chi = (s^\dagger\tilde{d} + d^\dagger s) + \chi(d^\dagger\tilde{d})^{(2)}$, N_B is the number of valence bosons, and c is a scaling factor. The above Hamiltonian contains two parameters, ζ and χ, with the parameter ζ ranging from 0 to 1, and the parameter χ ranging from 0 to $-\sqrt{7}/2 = -1.32$. With this parameterization, the entire symmetry triangle of the IBA, shown in Figure 2, can be described, along with each of the three dynamical symmetry limits of the IBA. The parameters (ζ,χ) can be plotted in the symmetry triangle by converting them into polar coordinates [15]

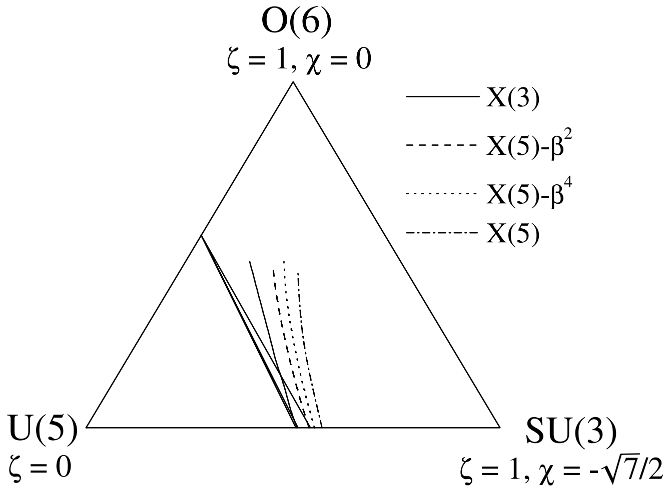

Figure **2.** IBA symmetry triangle illustrating the dynamical symmetry limits and their corresponding parameters. The phase transition region of the IBA, bordered by ζ^* on the left and by ζ^{**} on the right, as well as the loci of parameters which reproduce the $R_{4/2}$ ratios of X(3) (2.44), X(5)-β^2 (2.65) [24], X(5)-β^4 (2.77) [24], and X(5) (2.90) [2] are shown for $N_B = 10$. The line defined by $\zeta_{\rm crit}$ is also shown, lying to the right of the left border and almost coinciding with it.

$$\rho = \frac{\sqrt{3}\zeta}{\sqrt{3}\cos\theta_\chi - \sin\theta_\chi}, \qquad \theta = \frac{\pi}{3} + \theta_\chi, \tag{24}$$

where $\theta_\chi = (2/\sqrt{7})\chi(\pi/3)$.

Using the coherent state formalism of the IBA [11, 16, 17] one can obtain the scaled total energy, $E(\beta,\gamma)/(cN_B)$, in the form [18]

$$\mathcal{E}(\beta,\gamma) = \frac{\beta^2}{1+\beta^2}\left[(1-\zeta) - (\chi^2+1)\frac{\zeta}{4N_B}\right] - \frac{5\zeta}{4N_B(1+\beta^2)}$$
$$-\frac{\zeta(N_B-1)}{4N_B(1+\beta^2)^2}\left[4\beta^2 - 4\sqrt{\frac{2}{7}}\chi\beta^3\cos 3\gamma + \frac{2}{7}\chi^2\beta^4\right], \tag{25}$$

where β and γ are the two classical coordinates, related [11] to the Bohr geometrical variables [7].

As a function of ζ, a shape/phase coexistence region [19] begins when a deformed minimum, determined from the condition $\frac{\partial^2\mathcal{E}}{\partial\beta^2}|_{\beta_0\neq 0} = 0$, appears in addition to the spherical minimum and ends when only the deformed minimum remains. The latter is achieved when $\mathcal{E}(\beta,\gamma)$ becomes flat at $\beta = 0$, fulfilling the condition [14] $\frac{\partial^2\mathcal{E}}{\partial\beta^2}|_{\beta=0} = 0$, which is satisfied for

$$\zeta^{**} = \frac{4N_B}{8N_B + \chi^2 - 8}. \tag{26}$$

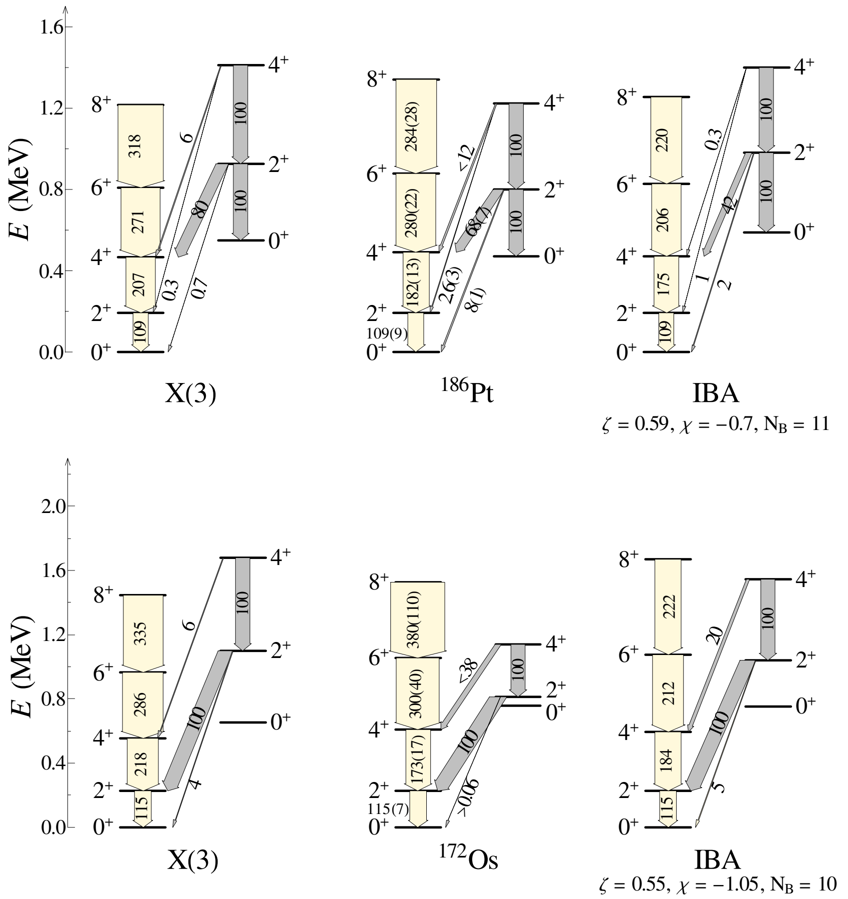

Figure **3.** Comparison of the experimental data (middle) to the X(3) predictions (left) and IBA calculations (right) for ^{186}Pt (top) and ^{172}Os (bottom). The thicknesses of the arrows indicate the relative (gray arrows) and absolute (white arrows) $B(E2)$ strengths which are also labelled by their values. The absolute $B(E2)$ strengths are normalized to the experimental $B(E2; 2_1^+ \rightarrow 0_1^+)$ value in each nucleus. Experimental data taken from Refs. [21–23].

In between there is a point, $\zeta_{\rm crit}$, where the two minima are equal and the first derivative of $\mathcal{E}_{min}$, $\partial\mathcal{E}_{min}/\partial\zeta$, is discontinuous, indicating a first-order phase transition. This point is [20]

$$\zeta_{\rm crit} = \frac{16N_B}{34N_B - 27}. \tag{27}$$

The range of ζ corresponding to the region of shape/phase coexistence shrinks with decreasing $|\chi|$ and converges to a single point for $\chi = 0$, which is the point of a second-order phase transition between U(5) and O(6), located on the U(5)–O(6) leg of the symmetry triangle (which is characterized by $\chi = 0$) at $\zeta = N_B/(2N_B - 2)$, as seen from Eq. (26). The phase transition region of the IBA is included in Figure 2.

In Figure 2 it is clear that the line reproducing the $R_{4/2}$ ratio of the X(3) model lies within the phase transition region of the IBA, or close to it. Nuclei exhibiting level schemes similar to X(3), namely ^{186}Pt and ^{172}Os, are shown in Figure 3, together with relevant IBA fits. In a similar manner one can see that ^{146}Ce and ^{174}Os provide good examples of the X(5)-β^2 model [24], while ^{158}Er and ^{176}Os are good examples of the X(5)-β^4 model [24].

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