Spectra of Odd Mass Nuclei within the Supersymmetric Extension of the Interacting Vector Boson Model

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Abstract. A supersymmetric extension of the dynamical symmetry group $Sp^B(12, R)$ of the symplectic Interacting Vector Boson Model /IVBM/, to the orthosymplectic group $OSp(4/12, R)$ is developed in order to incorporate fermion degrees of freedom into nuclear dynamics and to encompasse the treatment of odd mass nuclei. The bosonic sector of the supergroup is used to describe the complex collective spectra of the neighboring even-even nuclei and is considered as a core structure of the odd nucleus. The fermionic sector is represented by the fermion spin group $SU^F(2) \subset O^F(4)$. The so obtained, new exactly solvable limiting case is applied for the description of the nuclear collective spectra of odd mass nuclei. The theoretical predictions for different collective bands in odd mass nuclei from the actinide region are compared with the experiment. The obtained results reveal the applicability of the models extension.

1 Introduction

Symmetry is an important concept in nuclear physics. In finite many-body systems of this type, it appears as time reversal, parity, and rotational invariance, but also in the form of dynamical symmetries.

Many collective properties of the nuclei have been investigated using models based on dynamical groups. One of the most popular and widely used models of this type are the Interacting Boson Model (IBM) [1] and its extensions [2, 3] as well as the symplectic model [4], based on the group $Sp(6, R)$. In them, one obtains bands of collective states which span irreducible representations of the corresponding dynamical groups and whose corresponding properties, such as energy levels and electromagnetic transition strengths, can be determined by algebraic methods.

It is well known that nucleons have intrinsic spins and that there are strong spin-orbit interactions. Moreover, the experiment has revealed, that the presence of spin does not prevent the appearance of rotational bands. It also established that the rotational character of the different collective bands is similar for neighboring even-even and odd-even nuclei far from closed shells. For the description of the nuclear spectra of such even-even nuclei the above mentioned variety of boson models is used. This is possible because in the even-even nuclei we consider pairs of nucleons coupled to integer angular momentum. However, this is not the case for odd mass nuclei. Thus, the following question naturally arises: how to incorporate fermion degrees of freedom into the nuclear dynamics in a way that the rotational character of the collective bands is preserved.

The natural extension of IBM, the Interacting Boson-Fermion Model (IBFM) [5], which includes single-particle (fermion) degrees of freedom in addition to the collective (boson) ones, have provided in the last decays a unified framework for the description of even-even and odd-even nuclei distant from closed shell configurations, at least in the low-angular momentum domain.

For the description of odd$-A$ nuclei, a fermion needs to be coupled to the N boson system. This can be done by a semimicroscopical approach which relies on seniority in the nuclear shell model [2]. As an alternative to this, in the IBFM approach, Hamiltonians exhibiting dynamical Bose-Fermi symmetries, that are analytically solvable [5] are constructed. Thus, the extension of the IBM for the case of odd mass nuclei leads to the group structure $U^B_\pi(6) \otimes U^B_\nu(6) \otimes U^F(m)$ (IBFM-2) or $U^B(6) \otimes U^F(m)$ (IBFM-1), where $m = \sum_j (2j+1)$ is the dimension of the single-particle space. Obviously, in the general case for arbitrary $m-$ values, analytical expressions for the nuclear levels would be too cumbersome and will contain too many parameters. Moreover, orbitals higher in energy than those of the valence shell might play a role and have to be included in the model, thus breaking the supersymmetric scheme. Therefore, numerical calculations have to be performed with schematic Hamiltonians. These deficiencies, motivate the development of the new extension of the IVBM, which will be based on the success of the boson description of the even-even nuclei, but will include the fermion degrees of freedom in a simple and straightforward way, that still leads to exact analytic solutions.

In the early 1980s, a boson-number-preserving version of the phenomenological algebraic Interacting Vector Boson Model (IVBM) [6] was introduced and applied successfully [7] to a description of the low-lying collective rotational spectra of the even-even medium and heavy mass nuclei. With the aim of extending these applications to incorporate new experimental data on states with higher spins and to incorporate new excited bands, we explored the symplectic extension of the IVBM [8], for which the dynamical symmetry group is $Sp(12, R)$. This extension is realized from, and has its physical interpretation over basis states of its maximal compact subgroup $U(6) \subset Sp(12, R)$, and resulted in the description of various excited bands of both positive and negative parity of complex systems exhibiting rotation-vibrational spectra. With the present work we extend the earlier applications of IVBM for the description of the ground and first excited positive and negative bands of odd mass nuclei. In order to do this we propose a new dynamical symmetry and apply it to the real odd nuclear systems.

Thus, in this paper a supersymmetric extension of the dynamical symmetry group $Sp^B(12, R)$ of the symplectic Interacting Vector Boson Model to the orthosymplectic group $OSp(4/12, R)$ is presented in order to incorporate fermion degrees of freedom into the nuclear dynamics and in doing so to be able to treat the spectra of odd mass nuclei. The bosonic sector (described by $Sp^B(12, R)$) of the supergroup is used to describe the complex collective spectra of the neighboring even-even nuclei and is considered as a core structure of the odd nucleus, while through its fermionic sector the fermion spin group $SU^F(2) \subset O^F(4)$ is involved into the algebraic considerations of the collective states of the odd nucleus. In this way, we present here a new exactly solvable limiting case, based on the reduction of $Sp^B(12,R)$ through its maximal compact subgroup $U^B(6)$. The theoretical predictions for different collective bands of both positive and negative parity for odd mass nuclei from actinide and rare earth region are compared with the experiment. The obtained results confirm the extended applicability of the model.

2 The Inclusion of Spin

In order to incorporate the intrinsic spin degrees of freedom into the symplectic IVBM, we extend the dynamical algebra of $Sp(12,R)$ to the orthosymplectic algebra of $OSp(4/12,R)$. For this purpose we introduce a particle (quasiparticle) with spin $S = 1/2\hbar$ and consider a simple core plus particle picture. Thus, in addition of the boson collective degrees of freedom (described by dynamical symmetry group $Sp(12,R)$) we introduce creation and annihilation operators a_j^+ and a_j, which satisfy the anticommutation relations

$$\{a_i^+, a_j^+\} = \{a_i, a_j\} = 0,$$
$$\{a_i, a_j^+\} = \delta_{ij}, \qquad i,j = 1,2 \tag{1}$$

All bilinear combinations of a_j^+ and a_j, namely

$$f_{ij} = a_i^+ a_j^+,$$
$$g_{ij} = a_i a_j, \tag{2}$$
$$C_{ij} = (a_i^+ a_j - a_j a_i^+)/2 \tag{3}$$

generate the (quasispin) Lie algebra of $O^F(4)$. The number preserving operators (3) generate maximal compact subalgebra of $O^F(4)$, i.e. $U^F(2)$.

It is known that the spin group is $SU^F(2)$, which can be easily obtained from $U^F(2)$ excluding the number operator N^F. From hereafter the upper script F or B is referred to boson or fermion degrees of freedom, respectively.

Noting that the following embedding exists $SU^F(2) \subset O^F(4)$, we make supersymmetric extension of the IVBM which is defined through the chain (4), where bellow the different subgroups the quantum numbers characterizing their irreducible representations are given. We will point out only that, when restricted to the group $U^B(6)$, each irrep of the group $Sp^B(12,R)$ decomposes into irreps of the subgroup characterized by the totaly symmetric representations [8,9] $[N,0^5]_6 \equiv [N]_6$, where $N = 0,2,4,\ldots$ or $N = 1,3,5,\ldots$. We consider only the even (N-even) irreducible representation of $Sp^B(12,R)$ and will further omit the superscript B for the number of bosons N. The reduction of the latter to the rotational group $SO(3)$ through its compact subgroup $U^B(6)$ is given in details in [8], so we will not consider it here.

$$
\begin{array}{llll}
OSp(4/12,R) \supset & O^F(4) & \otimes & Sp^B(12,R) \\
 & \Downarrow & & \Downarrow \\
 & U^F(2) & \otimes & U^B(6) \\
 & \Downarrow & & \Downarrow \\
 & SU^F(2) & \otimes & SU^B(3) \times U_T^B(2) \\
 & S & & (\lambda,\mu) \Longleftrightarrow (N,T) \\
 & & \searrow & \Downarrow \\
 & & \otimes & SO^B(3) \times U(1) \\
 & & & \quad L \qquad\quad T_0 \\
 & & \Downarrow & \\
 & Spin^{BF}(3) & \supset & Spin^{BF}(2), \\
 & \quad J & & \quad J_0
\end{array}
\tag{4}
$$

From (4) it can be seen that the coupling of the boson and fermion degrees of freedom is done on the level of the angular momenta.

3 The Basis States and Energy Spectrum

We can label the basis states according to the chain (4) as:

$$
\mid [N]_6; (N,T); KL; S; JJ_0; T_0 \rangle, \tag{5}
$$

where $[N]_6-$ is the $U(6)$ labeling quantum number, $(N,T)-$ are the $SU(3)$ quantum numbers characterizing the core excitations, K is the multiplicity index in the reduction $SU(3) \subset SO(3)$, L is the core angular momentum, $S-$ is the spin of the odd particle, J, J_0 are the total (coupled) angular momentum and its third projection, and T,T_0 are the pseudospin and its third projection, respectively.

The infinite set of basis states classified according to the reduction chain (4) are schematically shown in Table 1. The fourth and fifth columns show the $SO^B(3)$ content of the $SU^B(3)$ group, given by the standard Elliott's reduction rules [10], while in the next column are given the possible values of the common angular momentum J, obtained by coupling of the orbital momentum L with the spin S. In the final column are given the possible values for the quantum number $K_J = K \pm 1/2$, which are used to label the different collective bands. The detailed procedure for construction of the basis states (5) will be given in forthcoming paper [11].

The Hamiltonian can be written as linear combination of the Casimir operators of the different subgroups in (4):

$$
\begin{aligned}
H = {} & aN + bN^2 + \alpha_3 T^2 + \beta_3' L^2 + \alpha_1 T_0^2 \\
& + \eta S^2 + \gamma' J^2 + \zeta J_0^2
\end{aligned}
\tag{6}
$$

and it is obviously diagonal in the basis (5) labeled by the quantum numbers of their representations. Then the eigenvalues of the Hamiltonian (6), that yield the spectrum of the odd mass system are:

Table **1.** Classification scheme of basis states (5) according the decompositions given by the chain (4).

N	T	(λ,μ)	K	L	J	K_J
0	0	$(0,0)$	0	0	$1/2$	$1/2$
2	1	$(2,0)$	0	$0,2$	$1/2;\ 3/2,5/2$	$1/2$
	0	$(0,1)$	0	1	$1/2;\ 3/2$	$1/2$
4	2	$(4,0)$	0	$0,2,4$	$1/2;\ 3/2,5/2;\ 7/2,9/2$	$1/2$
	1	$(2,1)$	1	$1,2,3$	$1/2,3/2;\ 3/2,5/2;\ 5/2,7/2$	$1/2;3/2$
	0	$(0,2)$	0	$0,2$	$1/2;\ 3/2,5/2$	$1/2$
6	3	$(6,0)$	0	$0,2,4,6$	$1/2;\ 3/2,5/2;\ 7/2,9/2;$ $11/2,13/2$	$1/2$
	2	$(4,1)$	1	$1,2,3,4,5$	$1/2,3/2;\ 3/2,5/2;\ 5/2,7/2;$ $7/2,9/2;\ 9/2,11/2$	$1/2;3/2$
	1	$(2,2)$	2	$2,3,4$	$3/2,5/2;\ 5/2,7/2;\ 7/2,9/2$	$3/2;5/2$
			0	$0,2$	$1/2;\ 3/2,5/2$	$1/2$
	0	$(0,3)$	0	$1,3$	$1/2,3/2;\ 5/2,7/2$	$1/2$
8	4	$(8,0)$	0	$0,2,4,6,8$	$1/2;\ 3/2,5/2;\ 7/2,9/2;$ $11/2,13/2;\ 15/2,17/2$	$1/2$
	3	$(6,1)$	1	$1,2,3,4,5,6,7$	$1/2,3/2;\ 3/2,5/2;\ 5/2,7/2;$ $7/2,9/2;\ 9/2,11/2;$ $11/2,13/2;\ 13/2,15/2$	$1/2;3/2$
	2	$(4,2)$	2	$2,3,4,5,6$	$3/2,5/2;\ 5/2,7/2;\ 7/2,9/2;$ $9/2,11/2;\ 11/2,13/2$	$3/2;5/2$
			0	$0,2,4$	$1/2;\ 3/2,5/2;\ 7/2,9/2$	$1/2$
	1	$(2,3)$	2	$2,3,4,5$	$3/2,5/2;\ 5/2,7/2;\ 7/2,9/2;$ $9/2,11/2$	$3/2;5/2$
			0	$1,3$	$1/2,3/2;\ 5/2,7/2$	$1/2$
	0	$(0,4)$	0	$0,2,4$	$1/2;\ 3/2,5/2;\ 7/2,9/2$	$1/2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$E(N;T,T_0;L,S;J,J_0) = aN + bN^2 + \alpha_3 T(T+1) + \beta_3' L(L+1) + \alpha_1 T_0^2 + \eta S(S+1) + \gamma' J(J+1) + \zeta J_0^2. \tag{7}$$

We note that only the last three terms of (6) come from the supersymmetric extension. We choose parameters $\beta_3' = \frac{1}{2}\beta_3$ and $\gamma' = \frac{1}{2}\gamma$ instead of β_3 and γ in order to obtain the Hamiltonian form of ref. [8] (setting $\beta_3 = \gamma$), when for the case $S = 0$ (hence $J = L$) we recover the symplectic structure of the IVBM.

4 Application of the New Dynamical Symmetry

In this paper we expand the earlier application of the IVBM [8], developed for the description of the collective bands of even-even nuclei, in order to include in our considerations the case of odd mass nuclei.

The most important application of the $U^B(6) \subset Sp^B(12, R)$ limit of the theory, which is part of the reduction of (4), is the possibility it affords for describing both even and odd parity bands up to very high angular momentum. In order to do this we first have to identify the experimentally observed bands with the sequences of basis states from the Table 1. As we deal with the (ortho) symplectic extension of the boson representations of the number preserving $U^B(6)$ symmetry we are able to consider all even eigenvalues of the number of vector bosons N with the corresponding set of pseudospins T, which uniquely define the $SU^B(3)$ irreps (λ, μ) given in the third column of Table 1.

In the present application of the model several points are of importance. First, following [8] we define the parity of the states as $\pi = (-1)^T$. This allows us to describe both positive and negative parity bands. The latter requires only the proper choice of the core band heads, on which the corresponding odd$-A$ collective bands are developed. Our choice is motivated by the fact, which has been always understood in nuclear physics, that well defined rotational bands can exist only when they are adiabatic relative to other degrees of freedom. In this way (in the adiabatic approximation) the single particle motion follows the correlated collective one of all nucleons. This allows us to assume that after the coupling of particle (with $S = 1/2\hbar$) to the boson core (collective degrees of freedom) the resulted states will possess all the collective properties of the core, including the parity.

The second point is the use of the algebraic concept of yrast states, also introduced in [8]. According to this notion we consider as yrast states the states with given L, which minimize the energy (7) with respect to the number of vector bosons N that build them. In the present considerations the yrast conditions yield relations between the number of bosons N and the coupled angular momentum J that characterizes each collective state. By means of it and taking into account the parity we can write the following relations if the ground state band (GSB) of the odd mass nuclei is:

$$\begin{aligned} K^\pi &= \tfrac{1}{2}^+, \qquad N = 2(J - \tfrac{1}{2}) \\ K^\pi &= \tfrac{7}{2}^+, \qquad N = 2(J - \tfrac{1}{2}) + 4 \\ K^\pi &= \tfrac{1}{2}^-, \qquad N = 2(J + \tfrac{1}{2}). \end{aligned} \tag{8}$$

From (8) one can see that to each of the states with given angular momentum J corresponds a definite number of bosons N. So for the description of the GSB, for example $K^\pi = \frac{1}{2}^+$, we use the sequence of states with different numbers of bosons $N = 2(J - \frac{1}{2}) = 0, 2, 4, \ldots$ for the corresponding values $J = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ and pseudospin $T = 0$ ($T_0 = 0$) of Table 1.

The final point is related to the description of the excited bands. It was established [12], that for the proper reproduction of the collective behavior of the different

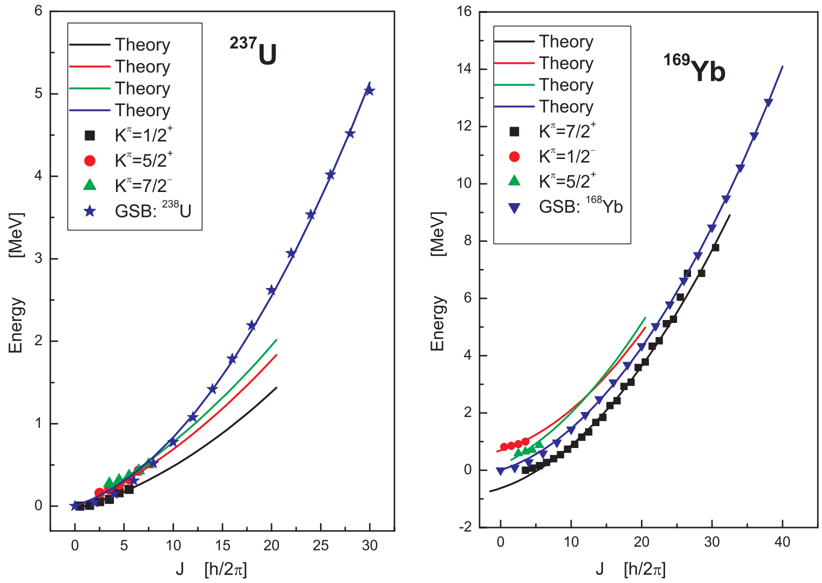

Figure **1.** Comparison of the theoretical and experimental energies for ground and excited bands of ^{237}U (left) and ^{169}Yb (right).

bands their band head structure has to be taken into account which in our case is determined by the minimal or initial number of bosons N_i. The latter determines the starting position of each excited band.

Thus, for the description of the different excited bands, we first determine the N_i of the band head structure and then we use the chosen correspondence between the states of the corresponding band to the sequence of basis states with $N = N_i, N_i + 2, N_i + 4, \ldots$ ($\Delta N = 2$) and $T = even = fixed$ or $T = odd = fixed$ for positive or negative parity, respectively. We will point out that the (ortho) symplectic structure of the model space give us rather rich possibilities to map basis states onto experimentally observed ones. Thus, another possibility of developing the sequence of basis states is to take again $N = N_i, N_i + 2, N_i + 4, \ldots$ ($\Delta N = 2$) but to change $T = T_i, T_i + 2, T_i + 4, \ldots$ ($\Delta T = 2$) in such a way, that the parity of band states is preserved even or odd, respectively.

Putting all pieces together, we apply our model considerations for the description of the nuclear spectra of four odd mass nuclei from the rare earth and actinide region. The phenomenological model parameters $a, b, \alpha_3, \beta_3, \eta, \gamma$ and ζ are evaluated by a fit to the experimental data [13]. The comparison between the experimental spectra and our calculations using the values of the model parameters given in Table 2 for the nuclei ^{237}U, ^{169}Yb, ^{173}Hf and ^{181}Pt is illustrated in Figures 1 and 2. In Table 2 the values of N_i, pseudospin T, orbital momentum L and the number of experimental states s are also given. From it one can see that in addition to the ground state band, the first excited bands with positive and negative parity are also

Table **2.** Values of the parameters of the model Hamiltonian (6) obtained in the fitting to the experimental spectra of the considered nuclei.

Nucl. bands	s	T	N	L	N_i	χ^2	parameters
^{237}U							
GSB: ^{238}U	16	0	$2L$		0	0.0063	$a = 0.01686$
$(K^\pi = 0^+)$							$b = -0.00034$
$K^\pi_{gb} = 1/2^+$	6	0	$2(J - \frac{1}{2})$	$J - \frac{1}{2}$	0	0.0002	$\alpha_3 = 0.00985$
$K^\pi = 5/2^+$	5	2	$2(J - \frac{1}{2})$	$J - \frac{1}{2}$	4	0.0002	$\beta_3 = 0.00576$
$K^\pi = 7/2^-$	5	3	$2(J - \frac{1}{2}) + 14$	$J - \frac{1}{2}$	20	0.00005	$\eta = -0.02998$
							$\gamma = 0.00497$
							$\zeta = -0.01318$
^{169}Yb							
GSB: ^{168}Yb	20	0	$2L$		0	0.0054	$a = 0.03662$
$(K^\pi = 0^+)$							$b = 0.00088$
$K^\pi_{gb} = 7/2^+$	26	2	$2(J - \frac{1}{2}) + 4$	$J - \frac{1}{2}$	10	0.0385	$\alpha_3 = -0.06185$
$K^\pi = 1/2^-$	4	1	$2(J + \frac{1}{2})$	$J - \frac{1}{2}$	2	0.0029	$\beta_3 = 0.00338$
$K^\pi = 5/2^+$	4	4	$2(J + \frac{1}{2}) + 12$	$J - \frac{1}{2}$	18	0.0178	$\eta = -0.49477$
							$\gamma = 0.00241$
							$\zeta = 4.5990$
^{173}Hf							
GSB: ^{174}Hf	9	0	$2L$		0	0.0055	$a = 0.016629$
$(K^\pi = 0^+)$							$b = -0.00128$
$K^\pi_{gb} = 1/2^-$	17	1	$2(J + \frac{1}{2})$	$J \mp \frac{1}{2}$	2	0.0005	$\alpha_3 = -0.01710$
							$\beta_3 = 0.01488$
							$\eta = -0.04559$
							$\gamma = 0.01377$
^{181}Pt							
GSB: ^{182}Pt	9	0	$2L$		0	0.0036	$a = 0.03351$
$(K^\pi = 0^+)$							$b = -0.00124$
$K^\pi_{gb} = 1/2^-$	17	1	$2(J + \frac{1}{2})$	$J \mp \frac{1}{2}$	2	0.0021	$\alpha_3 = -0.03937$
							$\beta_3 = 0.01359$
							$\eta = -0.010498$
							$\gamma = 0.01207$

included. From Table 2, where χ^2 is also given, and Figures 1 and 2 one can see the good agreement between the theoretical predictions and the experimental data. In the Figures 1 and 2, for comparison, the ground state band of the neighboring even-even nucleus is given too, described by the same set of model parameters.

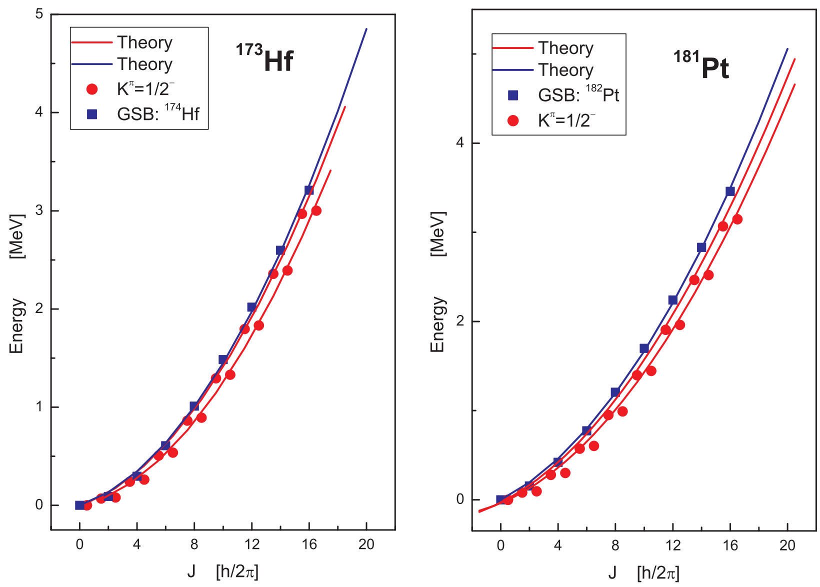

Figure **2.** Comparison of the theoretical and experimental energies for ground bands of ^{173}Hf (left) and ^{181}Pt (right).

Two interesting examples are shown in Figure 2, from which it can be seen that the states of the ground band of the odd mass nucleus are grouped in almost degenerate doublets. Such doublets are easily described in the present approach, when the two members of a certain doublet are taken from the basis states, characterized by the same number of bosons N, i. e. to the experimentally observed states of the doublet we put into correspondence the basis states with consequent values of $J = L \pm 1/2$ and the same N: $\mid [N]_6; (N,T); KL; S; J-1, J_0'; T_0 \rangle$ and $\mid [N]_6; (N,T); KL; S; JJ_0; T_0 \rangle$. Such doublet splitting is an indicator for the presence of "pseudospin" (not to be confused with pseudospin of the vector bosons) symmetry.

5 Conclusions

In this work we extended the dynamical symmetry group $Sp(12,R)$ of the IVBM to the orthosymplectic one $OSp(4/12,R)$. We introduced the fermion degrees of freedom by means of including a particle (quasiparticle) with spin $S = 1/2\hbar$ and exploiting the corresponding reduction $O^F(4) \supset SU^F(2)$.

Further, the basis states of the odd systems are classified by the new dynamical (super)symmetry (4) and the model Hamiltonian is written in terms of the first and second order invariants of the groups from the corresponding reduction chain. Hence the problem is exactly solvable within the framework of the IVBM which, in turn, yields a simple and straightforward application to real nuclear systems.

We present results that were obtained through a phenomenological fit of the models' predictions for the spectra of collective states to the experimental data for odd$-A$ nuclei from the rare-earth and actinide major shells exhibiting rotational spectra. The good agreement between the theoretical and the experimental results confirms the applicability of the newly proposed dynamical symmetry of the IVBM.

The success is based first on the symplectic extension of the model which allow the mixing of the basic collective modes $-$rotational and vibrational ones. The supersymmetry group $OSp(4/12, R)$ which is natural generalization of the dynamical symmetry group $Sp(12, R)$ of the IVBM could be further used to examine the correlations between the spectroscopic properties of the neighboring even-even, odd-even and odd-odd spectra of the neighboring nuclei and the underlying supersymmetry which might be considered in nuclear physics as proved experimentally [14]. These investigations are the subject of the forthcoming paper, but our preliminary results presented in this work already suggest the possibility to obtain the typical signatures of the nuclear supersymmetry.

Acknowledgments

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References

1. F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge University Press, Cambridge, 1987).
2. F. Iachello and O. Scholten, *Phys. Rev. Lett.* **43**, 679 (1979).
3. F. Iachello, *Phys. Rev. Lett.* **44**, 772 (1980).
4. G. Rosensteel and D.J. Rowe, *Phys. Rev. Lett.* **38**, 10 (1977); *Ann. Phys.* **126**, 343 (1980); D. J. Rowe, *Rep. Prog. Phys.* **48**, 1419 (1985).
5. F. Iachello and P. Van Isacker, *The Interacting Boson Fermion Model* (Cambridge University Press, Cambridge, 1991).
6. A. Georgieva, P. Raychev, and R. Roussev, *J. Phys. G: Nucl. Phys.* **8**, 1377 (1982).
7. A. Georgieva, P. Raychev, and R. Roussev, *J. Phys. G: Nucl. Phys.* **9**, 521 (1983).
8. H. Ganev, V. Garistov, and A. Georgieva, *Phys. Rev. C* **69**, 014305 (2004).
9. C. Quesne, *J. Math. Phys.* **14**, 366 (1973).
10. J. P. Elliott, *Proc. R. Soc. A* **245**, 128, 562 (1958).
11. H. Ganev and A. Georgieva, to be published.
12. H. G. Ganev, A. I. Georgieva, and J. P. Draayer, *Phys. Rev. C* **72**, 014314 (2005).
13. Mitsuo Sakai, *Atomic Data and Nuclear Data Tables* **31**, 399 (1984); Level Retrieval Parameters, `http://iaeand.iaea.or.at/nudat/levform.html`.
14. F. Iachello, *Phys. Rev Lett.* **44**, 772 (1980); A. B. Balantekin, I. Bars, R. Bijker, and F. Iachello, *Phys. Rev. C* **27**, 1761 (1983); P. Van Isacker, J. Jolie, and K. Heyde, *Phys. Rev Lett.* **54**, 653 (1985); A. Metz *et. al.*, *Phys. Rev Lett.* **83**, 1542 (1999); J. Barea, R. Bijker, and A. Frank, *J. Phys. A: Math. Gen.* **37**, 10251 (2004).