Chaos and $1/f$ Noise in Quantum Systems*

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Abstract. The energy spectrum fluctuations of quantum systems can be formally considered as a discrete time series. The power spectrum of such signal gives rise to $1/f$ noise in chaotic systems, whereas integrable systems present $1/f^2$ noise. This statistic is very useful for the analysis of two kinds of imperfect spectra: those with missing levels whose number and position are unknown, and those in which all the quantum numbers cannot be determined and levels of different symmetries are mixed. The power spectrum analysis provides an estimation of the number of symmetries and the fraction of mixing and missing levels.

1 Introduction

During the last past two decades, spectral statistical analysis has become the main tool for the study of quantum chaos. It is now well established, through numerical simulations and the analysis of experimental data, that the spectral statistics of systems whose classical analog is chaotic follow the predictions of Random Matrix Theory (RMT) [1]. When the classical analog is regular, the statistical properties of the spectra are identical to the properties of a sequence of uncorrelated random numbers, and follow Poisson statistics [2]. Consequently, a quantum system is considered chaotic if the statistical properties of its spectrum coincide with those of RMT.

In this work, we present an alternative way to study the spectral fluctuations of quantum systems. Using an appropiate statistic, called δ_q, the sequence of energy levels can be viewed as a time series, and it can be studied by means of traditional methods borrowed from this discipline. In particular, it has been shown [3] that chaotic quantum systems are characterized by $1/f$ noise and integrable quantum systems by $1/f^2$ noise. The full functional forms of the power spectrum were derived for the classical RMT ensembles [4]. The analysis of this new statistic has been shown to be extremely useful in different situations by a number of authors [5–11].

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As an application of this formalism, we study what happens if we deal with spectra with missing levels or several mixed symmetries. The change in the spectral properties of sequencies with missing levels has been studied since long ago [12]. Results for the number variance statistic Σ^2 were already reported in [13]; very recently, Bohigas and Pato studied how the next-neighbor spacing distribution $P(s)$ behave when only a fraction F_o of the levels is detected [14]. Also the next-neighbor spacing distribution $P(s)$ was studied for spectra involving several symmetries [15, 16]; the long-range correlations for this case were derived by Guhr *et al.* [17]. Here we show the behavior of δ_q statistic for these cases: we derive some theoretical results and we compare them with realistic nuclear spectra simulated through shell model calculations.

2 Unfolding and Definition of δ_q Statistic

The objective of every spectral statistical analysis is to characterize the properties of the fluctuations of the spectra. To tackle this task we assume that the density of states $\rho(E)$ can be separated into a smooth part and a fluctuating part,

$$\rho(E) = \bar{\rho}(E) + \tilde{\rho}(E), \quad (1)$$

where $\bar{\rho}(E)$ is the smooth part of the level density and $\tilde{\rho}(E)$ is the fluctuating part. The same separation can be defined for the accumulated level density $N(E)$, which measures the number of levels up to a certain energy E in the system

$$N(E) = \int_{-\infty}^{E} dE' \rho(E'). \quad (2)$$

Therefore, we can distinguish between a smooth and a fluctuating part,

$$N(E) = \bar{N}(E) + \tilde{N}(E). \quad (3)$$

The procedure used to extract the information in the fluctuating part of the spectra is called unfolding, and it consists in removing the secular behavior of the level density. It can be achieved transforming the actual spectrum $\{E_i\}$ in a dimensionless one $\{\epsilon_i\}$ by means of the map $\epsilon_i = \bar{N}(E_i)$.

To study the spectral fluctuations as a time series we use the δ_q statistic, which is defined as follows,

$$\delta_q = \epsilon_{q+1} - \epsilon_1 - q. \quad (4)$$

It represents the deviation of the excitation energy of the $(q + 1)$th unfolded level from its mean value. If we appropiately shift the ground state of the system, it also represents the accumulated level density fluctuations at $E = E_{q+1}$.

The δ_q statistic can be viewed as a time series if we establish an analogy between the index q, which represents the order in energy, and the time; see [3] for a complete discussion. One of the most simplest way to study the statistical properties of this time series is by means of the power power spectrum, *i. e.* the square modulus of its Fourier transform,

$$P_k^\delta = \frac{1}{N}\left|\sum_{q=0}^{N-1}\delta_q \exp\left(-2\pi qk/N\right)\right|^2, \tag{5}$$

where N is the number of levels of the sequence. We use this tool to analyze spectral fluctuations; a more sophisticated analysis, involving high order momenta, is proposed in [18].

3 $1/f$ Noise as a Fingerprint of Quantum Chaos

3.1 Numerical Results

As it was pointed in the introduction, RMT can be considered the paradigmatic model of quantum chaos [17]. Therefore, a simple and generic way to determine the behavior of δ_q statistic for integrable and chaotic systems consists in studying the power spectrum P_k^δ of random matrix spectra. This theory deals with three basic Hamiltonian matrix ensembles that describe the spectra of chaotic systems: the Gaussian orthogonal ensemble (GOE) of N-dimensional matrices, applicable for time-reversal invariant systems either with rotational symmetry or with integer spin when the rotational symmetry is broken; the Gaussian unitary ensemble (GUE), applicable for systems without time-reversal symmetry; and the Gaussian symplectic ensemble (GSE), applicable for time-reversal invariant systems with half-integer spin and broken rotational symmetry. There are other more complex ensembles like deformed ensembles, band matrix ensembles, etc., but they will not be considered in this work. Instead, we include an ensemble of diagonal matrices whose elements are random Gaussian variables; we call it the *Gaussian diagonal ensemble* (GDE).

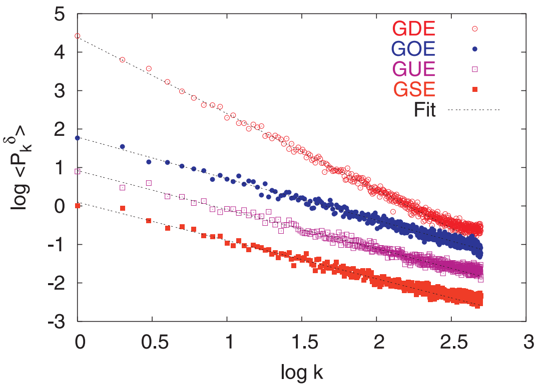


Figure **1.** P_k^δ for GOE, GUE, GSE and GDE spectra. The plots are displaced in the vertical axes to avoid overlapping.

Figure 1 shows the average value of P_k^δ for the random matrix ensembles introduced above, using a double logaritmic scale. In all the cases we have used matrices of dimension $N = 1000$, and we have averaged over 30 different matrices of each ensemble. It is clearly seen that the four cases are well described by means a power-law $P_k^\delta \propto k^{-\alpha}$, at least in the low and mid frequency region. A least-squares fit gives rise to the following results: $\alpha_{GDE} \approx 2$, and $\alpha_{GOE} \approx \alpha_{GUE} \approx \alpha_{GSE} \approx 1$. Chaotic systems are thus characterized by a $1/f$ noise, whereas integrable ones display a $1/f^2$ noise.

3.2 Theoretical Results

It has been shown [4] that the ensemble average of the δ_q power spectrum $\left\langle P_k^\delta \right\rangle$ can be written in a compact form as

$$\left\langle P_k^\delta \right\rangle = \frac{N^2}{4\pi^2}\left[\frac{K\left(k/N\right)-1}{k^2} + \frac{K\left(1-k/N\right)-1}{(N-k)^2}\right] + \frac{1}{4\sin^2\left(\dfrac{\pi k}{N}\right)} + \Delta. \quad (6)$$

This result is valid for $N \gg 1$, $k \in [1, 2, \ldots, N-1]$ and $\beta = 0, 1, 2$ and 4, that is, for the four ensembles we are taking into account. The function $K(\tau)$ is the so called spectral form factor

$$K(\tau) = \left\langle \left| \int d\epsilon \widetilde{\rho}(\epsilon) e^{-i2\pi\epsilon\tau} \right|^2 \right\rangle. \quad (7)$$

This magnitude is well known for GOE, GUE, GSE and GDE [16]. It can be obtained from the n-point correlation function, which gives the probability that n eigenvalues take the values $E_1, E_2, \ldots, E_n$ irrespective of the positions of the rest of the eigenvalues

$$R_n(\epsilon_1, ..., \epsilon_n) = \frac{N!}{(N-n)!}\int_{-\infty}^{\infty} d\epsilon_{n+1} ... \int_{-\infty}^{\infty} d\epsilon_N P_N(\epsilon_1, ..., \epsilon_N), \quad (8)$$

where $P_N(\epsilon_1, ..., \epsilon_N)$ is the probability of finding the N eigenvalues in positions $\epsilon_1, \epsilon_2, ..., \epsilon_N$. For practical purposes, this quantity is usually written in terms of the n-body cluster function $Y_n(\epsilon_1, ..., \epsilon_n) = 1 - R_n(\epsilon_1, ..., \epsilon_n)$. The most importat case is $n = 2$, which, for traslationally invariant spectra, depends only on the difference between the energies, so we can write $Y_2(\epsilon_1, \epsilon_2) = Y_2(\epsilon_2 - \epsilon_1) = Y_2(\epsilon)$. The form factor can be obtained from this last quantity as follows [16]

$$K(\tau) = 1 - \int_{-\infty}^{\infty} d\epsilon Y_2(\epsilon) \exp(-i\epsilon t/\hbar). \quad (9)$$

The term Δ is a correction due to the discrete nature of δ_q. Some subtle mathematical details involved in the ensemble average procedure give rise to [13]

$$\langle \delta_q^2 \rangle - \left\langle \widetilde{N}(E_q)^2 \right\rangle = \frac{1}{3}\int_{-\infty}^{\infty} drds Y_3(0,r,s) - \frac{1}{2}\left(\int_{-\infty}^{\infty} dr Y_2(r)\right)^2, \quad (10)$$

for canonical ensembles and $q \gg 1$. Using this last equation, a straighforward deduction [4] allows us to obtain

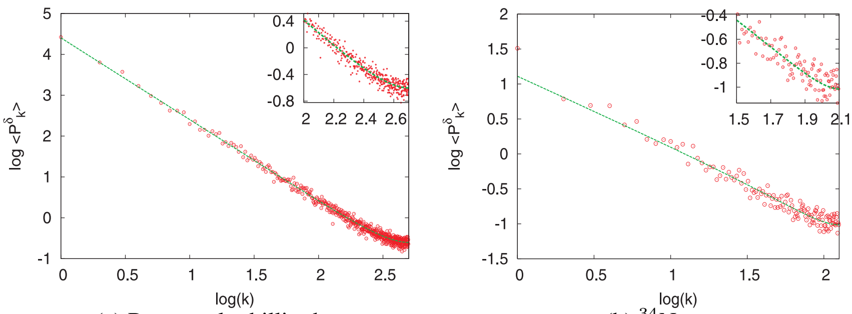
$$\Delta = \begin{cases} -\dfrac{1}{12} & \text{for Gaussian Ensembles,} \\ 0 & \text{for Poisson.} \end{cases} \quad (11)$$

The expresion (6) seems to be very complicated, far from the simple behavior numerically found for RMT. Nevertheless, if we perform a Taylor expansion of this equation, we find, for $k \ll N$ and $N \gg 1$,

$$\langle P_k^\delta \rangle = \begin{cases} \dfrac{N}{2\beta\pi^2 k} & \text{for chaotic systems,} \\ \dfrac{N^2}{4\pi^2 k^2} & \text{for integrable systems.} \end{cases} \quad (12)$$

That is, we reproduce the power-laws found numerically, with $\alpha = 1$ for chaotic systems, and $\alpha = 2$ for integrable ones.

To test the expresion (6), we compare its predictions with numerical calculations for different physical systems: a) a rectangular billiard in the high-energy region, as an example of integrable system, and b) a shell-model calculation for a very exotic nucleus, ^{34}Na. Figure 2 shows the results of this comparison. It is clearly seen that eq. (6) describes almost perfectly the numerical results in the whole frequency region. It is included a zoom of the high frequency region to show that the theoretical expression describes pretty well the deviations from the power-low behavior that appear in this region.



(a) Rectangular billiard (b) ^{34}Na

Figure **2.** P_k^δ for a rectangular billiard and for a shell-model calculation for ^{34}Na. Dashed lines represent the theoretical behavior predicted by (6).

4 Analysis of Imperfect Spectra

4.1 Theory

The theoretical framework developed in the last section can be also applied to the analysis of imperfect spectra, that is, with missing levels or mixed symmetries. Suppose that there are l complete, infinite and stationary level sequences with level densities $\rho_i(\epsilon)$, whose spectral fluctuations are given by an appropriate ensemble. The relevant parameters of the problem are the probability $\varphi_i(\epsilon_q)$ of observing one given level of the i-th sequence with energy ϵ_q, and the fractional densities $\eta_i = \overline{\rho_i}(\epsilon)/\left(\sum_{i=1}^{l} \overline{\rho_i}(\epsilon)\right)$. In order to make the statistical analysis tractable we consider two basic assumptions:

1. The probabilities $\varphi_i(\epsilon_q)$ are set equal to a constant φ_i for each sequence, meaning that the energy levels are dropped randomly and uniformly from the spectrum. From now on, these quantities will be called the fractions of observed levels.
2. The fractional densities are taken constant through the whole spectrum; therefore, except for a constant factor, the smooth shape of the different sequences is the same.

Lets consider first the case of a single incomplete level sequence. Assuming that the fraction of observed levels is $0 < \varphi < 1$, it can be shown that the n-point cluster functions are modified as [14]

$$\mathcal{Y}_n(\epsilon_1, ..., \epsilon_n) = Y_n(\epsilon_1/\varphi, ..., \epsilon_n/\varphi). \tag{13}$$

In what follows capital letters denote the statistical quanties of the actual spectrum, while "calligraphic" letters denote the same quantities for the observed spectrum.

From eq. (13) it is possible to calculate all the magnitudes involved in P_k^δ; the detalied calculations will be presented in a further paper [19]. The final result for the power spectrum of δ_q statistic as a function of φ is

$$\left\langle \mathcal{P}_k^\delta \right\rangle = \frac{N^2\varphi}{4\pi}\left[\frac{K\left(\frac{\varphi k}{N}\right)-1}{k^2} + \frac{K\left(\frac{\varphi(N-k)}{N}\right)-1}{(N-k)^2}\right] + \frac{1}{4\sin^2\left(\pi k/N\right)} - \varphi^2 \Delta. \tag{14}$$

Following a similar procedure, it is possible to calculate what happens if we consider the superposition of l level sequences with different quantum numbers and constant fractional densities η_i. Here we only show the main results; we refer the reader interested in detais to [19].

The average power spectrum in the general case is

$$\left\langle \mathcal{P}_k^\delta \right\rangle = \frac{N^2}{4\pi}\sum_{i=1}^{l}\eta_i\varphi_i\left[\frac{K_i\left(\frac{\varphi_i k}{N\eta_i}\right)-1}{k^2} + \frac{K_i\left(\frac{\varphi_i(N-k)}{N\eta_i}\right)-1}{(N-k)^2}\right] + \frac{1}{4\sin^2\left(\pi k/N\right)} + \left\langle\varphi\right\rangle^2 \Delta, \tag{15}$$

where $K_i(\tau)$ is the spectral form factor characteristic of the $i-th$ sequence and $\left\langle \varphi^2 \right\rangle = \sum_{i=1}^{l} \eta_i \left(\varphi_i\right)^2$.

Another particular case is that of complete sequences; writing down Eq. (15) when $\varphi_i = 1$, one obtains

$$\left\langle \mathcal{P}_k^\delta \right\rangle = \frac{N^2}{4\pi} \sum_{i=1}^{l} \eta_i \left[\frac{K_i\left(\frac{k}{N\eta_i}\right) - 1}{k^2} + \frac{K_i\left(\frac{N-k}{N\eta_i}\right) - 1}{(N-k)^2} \right] + \frac{1}{4\sin^2\left(\pi k/N\right)} + \Delta. \tag{16}$$

4.2 Numerics

In order to test the theoretical expression derived above, we have considered the sd nucleus ^{24}Mg, that has been used in refs. [3, 4] to show that the spectral fluctuations of quantum chaotic systems exhibit $1/f$ noise. We follow the standard shell model procedure to obtain a large enough number of energy levels. The construction and diagonalization of the JT Hamiltonian matrices was carried out by using the shell-model code NATHAN [20]. In this case, the valence space shells have positive parity, and therefore all the states have positive parity. We have considered only the cases with $T = 0$ and $J = 0, 1, 2, 3, 4, 5, 6, 7$ and 8.

Before we proceed to present the main results of this section, we describe briefly how the different types of sequences used in the statistical analysis are built:

1. *Mixed sequences*. All the levels from different J spectra are gathered together and ordered in increasing energy, regardless of their angular momentum; then the resulting sequence is unfolded. The mixed sequence is divided in several sets of $N = 256$ consecutive energy levels; the power spectrum of each set is calculated using a fast Fourier transform routine, and finally all these sets are used to compute some kind of "ensemble" average of the power spectrum. To improve this kind of average and clarify the main trend of the power spectrum, we divide the logarithmic frequency axis into equal bins and average locally the power spectrum components in each bin.
2. *Pure sequences*. After unfolding every J spectrum, we divide it in several sets of $N = 256$ consecutive levels. $\left\langle \mathcal{P}_k^\delta \right\rangle$ is calculated using the same procedure described above.
3. *Incomplete sequences.* To generate incomplete sequences (pure or mixed) we proceed as follows. Running along each sequence, the decision of keeping or dropping a given level ϵ_q is made by means of a random variable x uniformly distributed in the interval $[0, 1]$ and a smooth cut-off function $0 \leq \chi(\epsilon) \leq 1$ satisfying that $\sum_{q=1}^{N} \chi(\epsilon_q)/N = \varphi$. If $x > \chi(\epsilon_q)$ the level is dropped from the spectrum. With this procedure we drop roughly a fraction $(1-\varphi)$ of the levels, but this quantity can be slightly different from one sequence to another.

Figure 3 compares the predictions of Eq. (16) with the numerical values of $\left\langle \mathcal{P}_k^\delta \right\rangle$ calculated by using two pure sequences with $J = 3$ and $J = 4$ and a mixed sequence

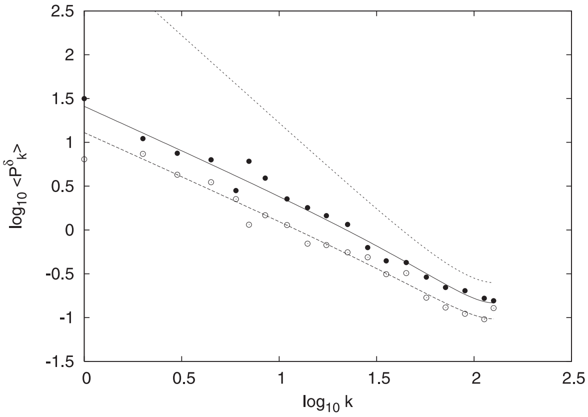


Figure **3.** Theoretical power spectrum of the δ_q function for GOE (dashed line) and Poisson (dotted line) and for the ensemble average of the superposition of two different GOE matrices (solid line) compared to numerical averages calculated by using $J = 3$ and $J = 4$ pure sequences of length $N = 256$ (open circles) and the superposition of these sequences (filled circles).

that was obtained superposing these two sequences. Since the dimensions of the two spectra are very similar, the fractional densities satisfy that $\eta_1 \approx \eta_2 \approx 0.5$. It is clearly seen that one can distinguish (at least in this case) the behavior of $\left\langle \mathcal{P}_k^\delta \right\rangle$ for pure and mixed sequences, and also that the agreement between the theoretical predictions and the numerical values is excellent.

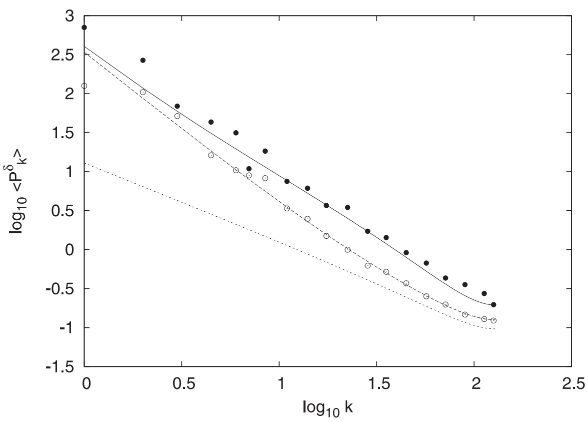


Figure **4.** Theoretical predictions of Eq. (15) for GOE (dotted line), GOE with a constant fraction of observed levels $\varphi = 0.8$ (dashed line) and the superposition of two GOE with constant $\varphi = 0.8$ (solid line). The theoretical values are compared with numerical averages using incomplete pure $J = 0 - 8$ sequences of length $N = 256$ (open circles), and imcomplete mixed sequences of the same length (filled circles).

To test if the theoretical expresions are also well suited to describe more involved situations, the average power spectrum of δ_q for nine incomplete sequences (with $J = 0-8$) and a mixed sequence is displayed in Figure 4. The fit of $\langle \mathcal{P}_k^\delta \rangle$ to the law (15) is excellent. Moreover, the main point is that the behavior of $\langle \mathcal{P}_k^\delta \rangle$ for pure and mixed sequences is clearly different along the whole frequency interval. Therefore the separate analysis of the power spectrum in the low and high frequency regions may provide us with information on the fraction of missing levels and on the number of symmetries present in the spectrum, respectively.

5 Conclusions

We have shown that the fluctuations of the energy spectra of quantum systems can be considered as time series. By means of an appropiate discrete statistic, δ_q, we have established that quantum chaotic systems are characterized by an almost ubiquitous property: the $1/f$ noise; moreover, quantum integrable systems present $1/f^2$ noise.

We have shown that the δ_q statistic can also be used to analyze spectra with missing levels or mixed symmetries. We have obtained theoretical expressions for the average δ_q power spectrum of spectra either with missing levels, mixed symmetries or missing levels as well as mixed symmetries. In order to test our predictions, we have used the standard shell model description of the chaotic nucleus ^{24}Mg. The main conclusion is that our procedure can be used to determine both the fraction of missing levels and the numer of symmetries involved in a spectrum. This is a very interesting and relevant result because very often it is difficult to measure all the levels in a given energy window and determine all the quantum numbers of every level.

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