Anharmonic Wobbling Motion

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Abstract. Wobbling motion of a nucleus is induced by triaxial deformation. Quantized wobbling motion, that is, wobbling phonon, is expected to show a harmonic spectrum according to the Bohr-Mottelson model. However, observed wobbling phonon spectrum shows a strong anharmonicity. In this paper, the sources for this anharmonicity are discussed through several models.

1 Introduction

It is no doubt that a nucleus is a complicated many-body system, but its collective modes can be sometimes understood in a surprisingly simple manner. An example is seen in a sequence of the low-lying excitations connected through strong E2 transitions. This spectrum corresponds to nuclear collective rotation, which can be approximated as a uniform one-dimensional rotation of a "rigid" body with an axial symmetry.

Nuclear shape plays an important role in providing various "flavours" to nuclear rotations. In studying rotational motion of a nucleus, it is convenient to classify nuclear deformation with multipolarity. For a deformed shape, the nuclear radius $R(\theta\phi)$ is orientation-dependent and thus expressed as

$$R(\theta\phi) = R_0 \left(1 + \sum_{lm} \alpha_{lm} Y_{lm}(\theta\phi) \right), \tag{1}$$

where ϕ and θ are the azimuthal and polar angles in the intrinsic coordinate frame, respectively. The average nuclear radius R_0 is given as $R_0 = 1.2A^{1/3}$ (fm) with A being the mass number. The leading order is the quadrupole ($l = 2$), and a truncation at this order brings a five-dimensional parameter space spanned by the following deformation parameters: $\alpha_{20}, \alpha_{2\pm1}, \alpha_{2\pm2}$.

These five-dimensional parameter space can be decomposed into the $3 + 2$ subspaces, that is, the three Euler angles $(\theta_1, \theta_2, \theta_3)$ and the two Hill-Wheeler coordinates (β, γ). The Euler angles describe collective rotation of the quadrupole-deformed nucleus, while the Hill-Wheeler coordinates correspond to the intrinsic deformation parameters in the rotating frame. Parameters β and γ carry information on elongation and triaxiality of the deformed nucleus, respectively.

2 Triaxial deformation and Wobbling motion

In this paper, we would like to focus on *triaxial quadrupole deformation* (or γ deformation) and its high-spin states. It was demonstrated by Bohr and Mottelson that triaxial deformation can induce nuclear *wobbling motion* [1]. With a rigid-rotor model, they incorporated triaxiality in the following Hamiltonian.

$$H = \sum_{i=1}^{3} \frac{I_i^2}{2\Im_i}, \tag{2}$$

where moment of inertia is set as

$$\Im_1 > \Im_2 > \Im_3. \tag{3}$$

For given energy, the above equation gives an ellipsoid called the *Binet ellipsoid* [2]. At the same time, the angular momentum conservation gives a sphere, $\sum_{i=1}^{3} I_i^2 = I^2$. Thus the motion of the angular momentum vector is determined as an intersection of the Binet ellipsoid and the angular momentum sphere. For a triaxial shape, the intersection shows a deviation from a circular motion (precession). This deviation is in fact a wobbling motion in the *classical mechanics*, where all the three components of the angular momentum vector can be specified.

In quantum mechanics, the angular momentum algebra in the intrinsic frame $[\hat{I}_i, \hat{I}_j] = -i\hbar \hat{I}_k$ (i, j, k cyclic) does not allow the simultaneous specification of the three components. This quantum effect also brings an orientational fluctuation. In case of triaxiality, only the total angular momentum is conserved. That is, $[\hat{H}, \hat{I}^2] = 0$, where $\hat{I}^2 = \sum_{k=1}^{3} \hat{I}_k^2$, while $[\hat{H}, \hat{I}_i] = \frac{i}{2}\left(\frac{1}{\Im_j} - \frac{1}{\Im_k}\right)(2\Im_j\Im_k + i\Im_j) \neq 0$. Inevitably, a quantized wobbling state is written as a superposition,

$$|\Psi^I_{\text{wbl}}\rangle = \sum_K C_K |IK\rangle, \tag{4}$$

where the basis in the right-hand side are obtained from the eigenvalue equations

$$\hat{I}_1 |IK\rangle = K|IK\rangle \tag{5}$$

$$\hat{I}^2 |IK\rangle = I(I+1)|IK\rangle. \tag{6}$$

It is known that the summation for K needs to be taken only for even (odd) K for positive (negative) signature, which is a discrete symmetry with respect to a π rotation around a principal axis of the triaxial rotor.

In the high-spin limit $I >> 1$, the rotational motion of a quantum triaxial rotor can be well approximated to be a classical motion, that is, the rotation is near one-dimensional and the quantum fluctuation causing the wobbling can be minimized. This means that the distribution $|C_K|^2$ has a sharp peak around $K = I$. In this case, the operator $\hat{I}_1$ can be replaced with a C-number I as a good approximation, and the Hamiltonian is then written as

$$\hat{H} = \frac{I(I+1)}{2\mathfrak{I}_1} + \hat{H}_{\text{wbl}}, \tag{7}$$

where

$$\hat{H}_{\text{wbl}} = \frac{1}{2}\left(\frac{1}{\mathfrak{I}_2} - \frac{1}{\mathfrak{I}_1}\right)\hat{I}_2^2 + \frac{1}{2}\left(\frac{1}{\mathfrak{I}_3} - \frac{1}{\mathfrak{I}_1}\right)\hat{I}_3^2. \tag{8}$$

The wobbling phonon operator $a^\dagger \equiv \frac{1}{\sqrt{I}}(\hat{I}_2 + i\hat{I}_3)$, is introduced here. Expanding the Hamiltonian with this wobbling phonon operator gives rise to the so-called dangerous terms such as $a^\dagger a^\dagger$. Similar to the Bogoliubov transformation in the BCS theory, a proper canonical transformation $b^\dagger = xa^\dagger - ya$, where x and y are real numbers to satisfy $x^2 - y^2 = 1$, can "diagonalize" (that is, eliminate the dangerous terms) the Hamiltonian. With the number operator defined as $\hat{n} = b^\dagger b$, the wobbling Hamiltonian can be finally expressed as

$$\hat{H}_{\text{wbl}} = \hbar\omega_{\text{wbl}}\left(\hat{n} + \frac{1}{2}\right), \tag{9}$$

where the wobbling excitation energy is given as

$$\hbar\omega_{\text{wbl}} = \frac{I}{\mathfrak{I}_1}\sqrt{\frac{(\mathfrak{I}_2 - \mathfrak{I}_1)(\mathfrak{I}_3 - \mathfrak{I}_1)}{\mathfrak{I}_2\mathfrak{I}_3}}. \tag{10}$$

In this way, the wobbling spectrum $E(I, n)$ is characterized by two quantum numbers, that is, the total angular momentum I and the wobbling phonon number n.

3 Triaxiality as a dynamical variable

In the wobbling model, moment of inertia is treated as a constant. This means that the Hill-Wheeler coordinates (β, γ) are just parameters to describe the static quadrupole deformation. In a sense, the wobbling model assumes a nucleus to be a *rigid body* with a static triaxial deformation.

As Rainwater [3], Bohr [4], Hill and Wheeler [5] suggested, nuclei can be also regarded as a *liquid drop*, where the liquid is assumed to be incompressible and its flow is supposed to be irrotational. When the velocity field is denoted as $\boldsymbol{v}$, the irrotational condition is given as $\boldsymbol{\nabla} \times \boldsymbol{v} = 0$. Using a basic vector analysis formula, this equation implies an existence of the scalar field χ to satisfy an relation $\boldsymbol{v} = -\boldsymbol{\nabla}\chi$. The additional condition of incompressibility ($\partial\rho/\partial t = 0$ or equivalently $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$) brings a Laplace equation for χ, and the general solution with the boundary condition can be expressed as

$$\chi(\boldsymbol{r}) = \sum_{lm} \frac{R_0^{2-l}}{l} r^l \dot{\alpha}_{lm} Y_{lm}(\theta\phi). \tag{11}$$

The kinetic energy $T = \int \rho \boldsymbol{v}^2/2dV$ can be calculated with help of the scalar field χ, and it is divided into two terms. That is, $T = T_v + T_r$, where

$$T_r = \frac{1}{2}\sum_k \Im_k \omega_k^2 \tag{12}$$

$$T_v = \frac{1}{2}B_2\left(\dot{\beta}^2 + \beta^2\dot{\gamma}^2\right). \tag{13}$$

Here $B_2 = \rho_0 R_0^5/2$ is a mass parameter and the dynamical moment of inertia is given as

$$\Im_k = 4B_2\beta^2 \sin^2\left(\gamma - \frac{2\pi}{3}k\right). \tag{14}$$

The kinetic energy T_r describes a contribution coming from a rotational motion. However, it is important to note that its moment of inertia depends upon β and γ as *dynamical variables*, not mere parameters like in the wobbling model. As the other kinetic term T_v indicates, the surface oscillation needs to be considered through the β and γ degrees of freedom. Rotational motion and nuclear shape are thus interwinded with each other to produce a complicated dynamics.

4 Microscopic gamma-soft rotor model

In the observed wobbling phonon spectrum, a strong anharmonicity was observed [6, 7]. Some theoretical analyses were attempted [8, 9], but no clear answer has been given to account for this unexpected feature in the wobbling spectrum. Considering the previous irrotational-flow model, it seems important to treat the gamma degree of freedom (γ) in addition to the wobbling degrees of freedom (θ, ϕ).

A simple microscopic model is presented here for the aim to take into account the orientational and gamma degrees of freedom simultaneously, which I name *microscopic gamma-soft rotor model* or m-GSR. This model is basically a microscopic model based on the variational theory, in the spirit of the generator coordinate method [5, 10]. The GCM ansatz is given as a superposition of Slater determinants of deformed Nilsson single-particle states. That is, the ansatz is expressed as

$$|\Psi\rangle = \int f(\gamma\theta)|\mathrm{HF}(\gamma\theta)\rangle d\gamma d\theta, \tag{15}$$

where the Slater-determinant $|\mathrm{HF}\rangle$ is defined as

$$|\mathrm{HF}(\gamma\theta)\rangle = \prod_i^N c_i^\dagger(\gamma\theta)|0\rangle. \tag{16}$$

The deformed Nilsson states $c_i^\dagger(\gamma\theta)|0\rangle$ are obtained through diagonalizing the single-particle routhian

$$h(\gamma\theta) = h_0 + h_{\mathrm{def}}(\gamma;\beta) - \Omega(\cos\theta j_x + \sin\theta j_z). \tag{17}$$

The first term is the spherical Nilsson potential, where the oscillator frequency is given as $\hbar\omega = 41A^{-1/3}$ (MeV); the second term is the (quadrupole) deformed potential ; and the last part is the tilted (2D) cranking term. To concentrate on the role of the triaxial degree of freedom, the elongation parameter β is fixed in the present model. Also, the wobbling is confined in the $x-z$ plane for the sake of the simplicity.

The many-body Hamiltonian to be minimized is the following:

$$\hat{H} = \hat{H}_0 - \frac{1}{2}\kappa \sum_{\mu=-2}^{2} \hat{Q}_\mu^\dagger \hat{Q}_\mu, \tag{18}$$

where H_0 is the spherical Nilsson Hamiltonian and operator Q_μ is the μ-th component of the quadrupole operator. The second term corresponds to a two-body residual interaction. The pairing correlation is neglected in this model just for simplicity. The variational equation

$$\delta \frac{\langle \Psi | \hat{H} | \Psi \rangle}{\langle \Psi | \Psi \rangle} = 0, \tag{19}$$

is considered with the weight function $f(\gamma\theta)$ as a variational function, and the Hill-Wheeler equation is obtained as

$$\int d\gamma' d\theta' \left(\mathcal{H}(\gamma\theta, \gamma'\theta') - E_n \mathcal{N}(\gamma\theta, \gamma'\theta') \right) f_n(\gamma'\theta') = 0. \tag{20}$$

The energy and norm overlap kernels are

$$\mathcal{H}(\gamma\theta, \gamma'\theta') = \langle \mathrm{HF}(\gamma\theta) | \hat{H} | \mathrm{HF}(\gamma'\theta') \rangle, \tag{21}$$

$$\mathcal{N}(\gamma\theta, \gamma'\theta') = \langle \mathrm{HF}(\gamma\theta) | \mathrm{HF}(\gamma'\theta') \rangle. \tag{22}$$

The presence of the norm overlap kernels implies that the basis for the GCM, that is, $|\mathrm{HF}(\gamma\theta)\rangle$, are non-orthogonal basis. Because of this, the diagonalization of the Hill-Wheeler equation needs a special care. In the present study, we employ a new set of basis $\{u_\nu\}$ obtained from the diagonalization of the norm overlap kernel. Namely,

$$\int d\gamma' d\theta' \mathcal{N}(\gamma\theta, \gamma'\theta') u_\nu(\gamma'\theta') = n_\nu u_\nu(\gamma\theta). \tag{23}$$

Note that that the norm overlap kernel is a Hermitian in the parameter space (γ, θ), so that the new basis $u_n(\gamma\theta)$ forms an orthonormal basis. With this basis, the weight function is expanded in such a manner of

$$f(\gamma\theta) = \sum_{\nu, n_\nu \neq 0} \frac{g_\nu}{\sqrt{n_\nu}} u_\nu(\gamma\theta). \tag{24}$$

The expansion coefficients g_ν now correspond to new variational parameters. In this representation, the Hill-Wheeler equation reduces to an ordinary eigenvalue equation:

$$\sum_\nu \langle \nu | \hat{H} | \nu' \rangle g^i_{\nu'} = E^i g^i_\nu, \tag{25}$$

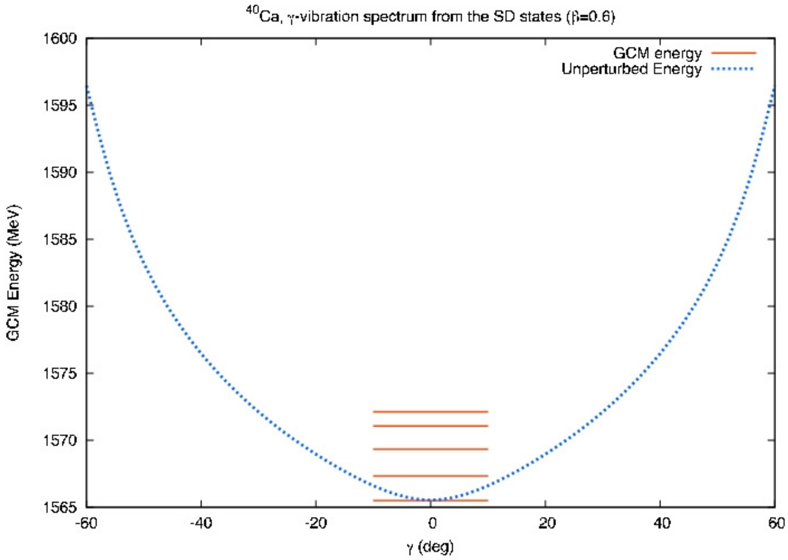

Figure 1. Gamma vibration spectrum at the no-cranked state ($\Omega = 0$).

where a new GCM ansatz is given as

$$|\nu\rangle = \frac{1}{\sqrt{n_\nu}} \int u_\nu(\gamma\theta)|\mathrm{HF}(\gamma\theta)\rangle d\gamma d\theta. \tag{26}$$

The matrix elements are calculated as

$$\langle\nu|\hat{H}|\nu'\rangle = \int\int\int\int d\gamma d\gamma' d\theta d\theta' \frac{u_\nu^*}{\sqrt{n_\nu}} \mathcal{H}(\gamma\theta, \gamma'\theta') \frac{u_{\nu'}}{\sqrt{n_{\nu'}}} \tag{27}$$

5 Numerical results

The goal of our calculation is to demonstrate the strong anharmonicity seen in the experimental data of the wobbling spectrum. The anharmonicity is seen already in the second phonon state, which is roughly expressed as

$$\frac{|E(I, n=2) - E(I, n=1)|}{|E(I, n=1) - E(I, n=0)|} \simeq \frac{1}{2}, \tag{28}$$

where n denotes a wobbling phonon number.

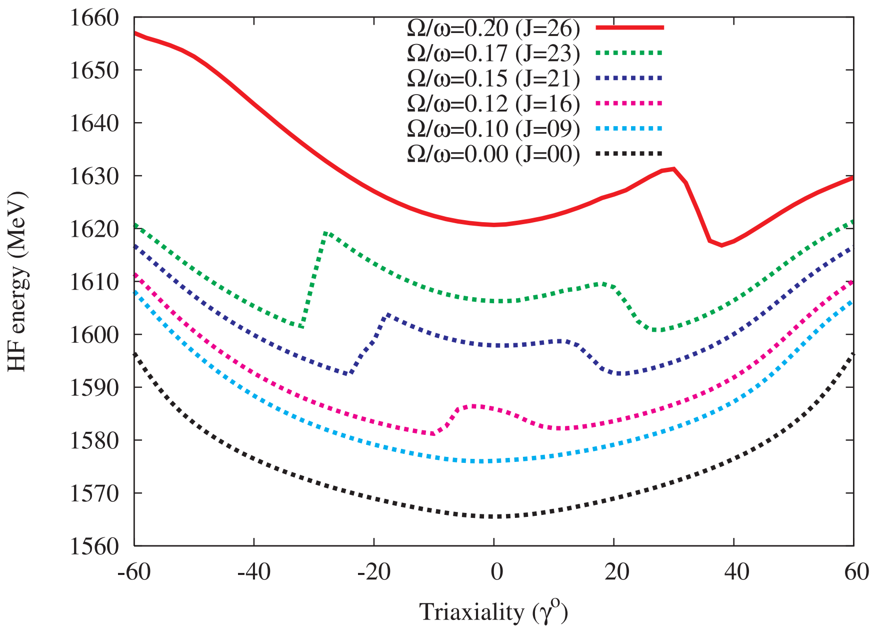

Figure 2. Gamma vibration spectrum of cranked states. Quantity Ω/ω means a ratio with respect to the harmonic oscillator frequency. Angular momentum values attached to Ω/ω are expectation values calculated only at $\gamma = 0$.

5.1 γ vibration

First of all, as a test calculation, only γ is taken as a dynamical variable (that is, generator coordinate in the GCM), to carry out the GCM calculation. The result is exhibited in Figure 1 for the no-cranked state ($\Omega = 0$). The discrete lines in the figure are the energy spectrum obtained through the GCM, while the dotted line attached to the graph is the expectation value

$$E(\gamma) = \langle \mathrm{HF}(\gamma)|\hat{H}|\mathrm{HF}(\gamma)\rangle = \mathcal{H}(\gamma, \gamma). \tag{29}$$

In other words, this quantity is diagonal part of the energy overlap kernels.

The numerical results show a slight anharmonicity, particularly in higher excited states. However, the lowest three levels maintain the harmonic structure to a good extent.

As the cranking starts, the energy surface starts to evolve (see Figure 2). The change is gradual up to $\Omega/\omega = 0.1$, where ω is the harmonic oscillator frequency. However, beyond this point, the energy curves show abrupt changes, which are results of configuration changes. This configuration changes happen due to the Ω cranking without a constraint on the total angular momentum, that is, $\langle \hat{J}_1 \rangle = J$. Therefore, the GCM calculation, which superposes these states, has little meaning without the constraint.

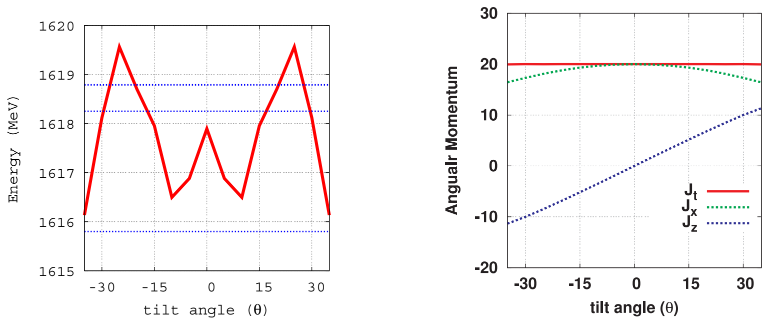

Figure 3. (Left) Wobbling spectrum for $\gamma = 40^\circ$ and $\langle \hat{J}_1 \rangle = 20\hbar$. (Right) The expectation values for each angular momentum component. These calculations are made with the angular momentum constraint in the GCM calculation.

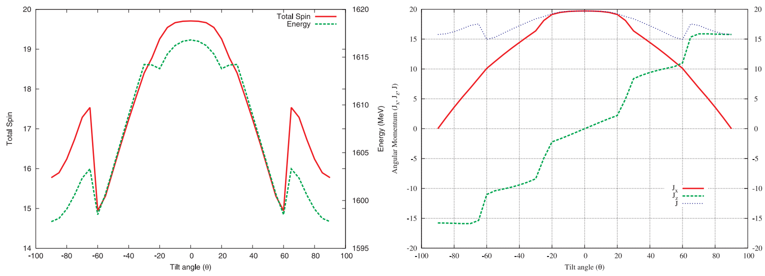

Figure 4. (Left) Expectations values of the total energy and total spin as a function of θ, without the angular momentum constraint. (Right) The expectation values for each angular momentum component, without the constraint.

5.2 θ wobbling

From Figure 2, the cranked state with $\Omega/\omega = 0.2$ has a triaxial minimum at $\gamma \simeq 40^\circ$. Picking up this solution, it is possible to construct the wobbling state (with the fixed γ value).

Energy curve for θ, that is, $E(\theta; \gamma = 40^\circ)$ and the corresponding GCM spectrum is displayed in Figure 3. Interestingly, the energy curve has a double-well structure, and the corresponding GCM spectrum gives rise to a strong anharmonicity already. The double-well potential in $E(\theta)$ implies an importance of tilted rotation. Tilted rotation breaks the signature symmetry spontaneously, but the symmetry allows the tilted solution $|\Psi(\theta)\rangle$ to be degenerate with its counterpart produced by the signature operation, that is, $|\Psi(-\theta)\rangle = \hat{R}_i(\pi)|\Psi(\theta)\rangle$. Quantum mechanical restoration of the symmetry thus happens through the quantum tunneling to allow the "wobbling" coupling mode between these two tilted rotating states. This mechanism to cause the anharmonicity was discussed by Matsuzaki and Ohtsubo [8]; and myself [9]. In this sense, the previous models for the anharmonicity were confirmed to be correct by the present microscopic model.

However, the current microscopic result needs a caution before the final conclusion is made, because due to a numerical difficulty to maintain the angular momentum constraint, the present GCM calculation can use the Slater determinant only up to $\theta \simeq \pm 35°$. If the full range in θ is taken into account, the GCM result may be different. Therefore, the result obtained here has to be regarded only as a provisional result, unfortunately. I am developing the effective algorithm to enable the full-range calculation at the moment.

In Figure 4, the cranking calculations without the angular momentum constraint are presented. In the left panel, the total energy curve is plotted, but its shape is very different from the one in Figure 3. There are several sudden jumps in the energy curve in Figure 4, which are caused by the artificial configuration change due to the loss of the constraint. As can be seen in the curve for the total angular momentum in Figure 4, the expectation value of the total angular momentum changes significantly away from the origin ($\theta = 0$). Consequently, the intrinsic structures become different and bring spurious structure changes. From this analysis, one can learn that the Ω-cranking calculations without the angular momentum constraint are not trustworthy, particularly in the tilted cranking calculations. Nevertheless, it is true that the constrained calculation requires more numerical efforts, and I am developing a new code to deal with this problem.

5.3 Full m-GSR calculations: $\gamma\theta$-GCM

Without the angular momentum constraint, the accuracy of the GCM calculation is doubtful. Therefore, the analysis of the full microscopic gamma-soft rotor model needs to wait for the progress in the code, which was mentioned above.

6 Conclusions

To understand the strong anharmonicity in the wobbling phonon spectrum, a microscopic gamma-soft rotor model is proposed. The model deals with the orientational fluctuation responsible for the wobbling degree of freedom (θ) as well as the surface oscillation coming from the dynamical triaxial degree of freedom (γ). Some test calculations were performed and it was found that the angular momentum constraint is very important to make the GCM calculation accurate and meaningful.

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