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Algebraic Approach to Non-central Potentials in the n-Dimensional Schrödinger Equation

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Abstract. The present work describes a method of solution of the n-dimensional Schrödinger equation with a potential group in higher dimensions. The method is applied to the algebraic solution of the Coulomb-Rosochatius system in n dimensions, whose bound states are connected with the potential group $SO(3n+1)$ and scattering states with $SO\left(3n,1\right)$. The S-matrix elements are computed by the method of intertwining operators and an integral representation is obtained for the scattering amplitude.

1 Introduction

In this work, we exploit the potential algebra of the Coulomb-Rosochatius [1] system in n dimensions

$$V(r) = -\frac{\gamma}{r} + \sum_{k=1}^{n} \frac{\beta_k}{x_k^2}, \tag{1}$$

in order to give a completely algebraic solution of bound states and scatttering states.

Before describing the method of solution, a few definitions are in order: a Lie group G is an invariance group for a quantum-mechanical system with Hamiltonian H if the latter can be related to a suitable function of a Casimir operator, C, of G

$$H = f\left(C\right) \ . \tag{2}$$

An algebraic derivation of the energy spectrum is possible also when the quantum-mechanical system admits a potential group. The concept of potential group, introduced in Ref. [2] and applied there to solvable one-dimensional potentials, relates the Hamiltonian, H, of a quantum-mechanical system to the projection of an appropriate function of a Casimir operator, C, of the group, G, to a subspace, $\mathcal{H}$, occurring in a reduction chain of the group

$$H = f\left(C\right)|_{\mathcal{H}} \ . \tag{3}$$

Formula (3) permits an algebraic derivation of the bound states of the system, which are labelled with the quantum numbers defining the unitary irreducible representations (unirreps) of the subgroups appearing in the reduction chain of the invariance group. The name of potential group is due to the fact that the quantum numbers of the unirreps of the final subgroup of the chain yield the coupling strengths of the potential in the Schrődinger equation.

As for scattering states, a fully algebraic derivation of the S matrix is possible when the Hamiltonian of the system satisfies an equation of type (2) or (3) with a non-compact group. It has been shown in Ref. [3] that the S matrix can be associated with an intertwining operator, A, between two Weyl-equivalent representations, U^{χ} and $U^{\widetilde{\chi}}$, of G, *i.e.* two representations of G with the same Casimir eigenvalues.

By definition, A satisfies the following equations

$$AU^{\chi}\left(g\right)=U^{\widetilde{\chi}}\left(g\right)A\ ,\ \forall\ g\ \epsilon\ G \tag{4}$$

and

$$AdU^{\chi}\left(b\right)=dU^{\widetilde{\chi}}\left(b\right)\ \ ,\ \ \forall\ b\ \epsilon\ \mathfrak{g} \tag{5}$$

where dU^{χ} and $dU^{\widetilde{\chi}}$ are the corresponding representations of the algebra, $\mathfrak{g}$, of G. Eqs. (4-5) have high restrictive power, determining the intertwining operator up to a constant. The S matrix coincides with the intertwining operator A

$$S=A \tag{6}$$

if eq. (2) holds, and with the reduction of A to a proper Hilbert subspace $\mathcal{H}$

$$S=A|_{\mathcal{H}} \tag{7}$$

if eq. (3) holds.

The plan of the paper is as follows: Section 2 will describe in full detail the potential group for the Coulomb-Rosochatius system, specialized to a compact group in Subsection 2.1, dedicated to bound states, and to a non-compact group in Subsection 2.2, dedicated to scattering states. Section 3 will be devoted to conclusions and perspectives.

2 Potential Group for the Coulomb-Rosochatius System

Let us start the discussion with the fact that the generators of the unitary irreducible representations (UIR's) of $SO\left(3n+1\right)$ (or $SO\left(3n,1\right)$) are Hermitian operators $M_{\mu\nu}=-M_{\nu\mu}$ ($\mu,\nu=1,2,\ldots,3n+1$) which obey the commutation relations

$$\left[M_{\mu\nu},M_{\sigma\lambda}\right]=i\left(g_{\mu\sigma}M_{\nu\lambda}+g_{\nu\lambda}M_{\mu\sigma}-g_{\mu\lambda}M_{\nu\sigma}-g_{\nu\sigma}M_{\mu\lambda}\right) \tag{8}$$

where

$$\begin{aligned} g_{\mu\nu} &= (+,+,\ldots,+,+) \quad \text{for} \quad SO\left(3n+1\right) \\ g_{\mu\nu} &= (+,+,\ldots,+,-) \quad \text{for} \quad SO\left(3n,1\right) \end{aligned} \tag{9}$$

The number of independent Casimir invariants which are identically multiple of the unit in each UIR is $\left[\frac{3n+1}{2}\right]$, where $[q]$ represents the largest integer contained in q.

Since the present work is dedicated to the motion of spinless particles, we can use the most degenerate (symmetric) UIR's of the group G of interest, where the eigenvalues of the Casimir invariants are identically zero, with the exception of the second order Casimir operator

$$C^G = \frac{1}{2}\sum_{\mu,\nu=1}^{3n+1} M_\mu^\nu M_\nu^\mu \ , \tag{10}$$

where G is $SO\,(3n+1)$ or $SO\,(3n,1)$.

It is well-known that the most degenerate representation of algebra $so(3n+1)$ ($so(3n,1)$) can be realized in the Hilbert space spanned by negative-energy (positive-energy) states corresponding to a fixed eigenvalue of the Coulomb Hamiltonian H^{Coul} in $3n$ dimensions, written in units $\hbar = m = 1$,

$$H^{Coul} = \frac{1}{2}p^2 - \frac{\gamma}{\sqrt{\zeta^2}}, \quad \gamma > 0 \tag{11}$$

where $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_{3n}) \in R^{3n}$, $p_j = -i\frac{\partial}{\partial\zeta_j}$, ($j = 1, ..., 3n$), $\zeta^2 = \sum_{i=1}^{3n} \zeta_i\zeta_i$, $p^2 = \sum_{i=1}^{3n} p_i p_i$.).

In order to exploit relation (3) we perform an unitary mapping in the Hilbert space of the system:

$$W: \quad \Psi^{Coul} \to \Phi = \lambda^{1/2}\left(\zeta\right)\Psi^{Coul} \tag{12}$$

implying the following similarity transformation of the generators

$$M_{ij} = \lambda^{1/2}\left(\zeta\right) \circ \left(\zeta_i p_j - \zeta_j p_i\right) \circ \lambda^{-1/2}\left(\zeta\right) \qquad (\, i, j = 1, ..., 3n) \tag{13}$$

$$\begin{aligned} M_{i,3n+1} &= |2h|^{-\frac{1}{2}}\,\lambda^{1/2}\left(\zeta\right) \circ \left[\zeta_i p^2 - p_i\left(\zeta\cdot p\right) + i\frac{3n-1}{2}p_i - \frac{\gamma\zeta_i}{\sqrt{\zeta^2}}\right] \\ &\circ \quad \lambda^{-1/2}\left(\zeta\right) = -M_{3n+1,i} \qquad (\, i = 1, ..., 3n) \end{aligned} \tag{14}$$

where

$$\lambda\left(\zeta\right) = \left(\zeta_1^2 + \zeta_2^2 + \zeta_3^2\right)\left(\zeta_4^2 + \zeta_5^2 + \zeta_6^2\right)\cdots\left(\zeta_{3n-2}^2 + \zeta_{3n-1}^2 + \zeta_{3n}^2\right) \tag{15}$$

and

$$h = \lambda^{1/2}\left(\zeta\right) \circ \left(\frac{1}{2}p^2 - \frac{\gamma}{\sqrt{\zeta^2}}\right) \circ \lambda^{-1/2}\left(\zeta\right). \tag{16}$$

It is worth pointing out that the M_{ij} generators (13) come from the components of the angular momentum, $L_{ij} = \zeta_i p_j - \zeta_j p_i$, and the $M_{i,3n+1}$ generators (14) from the components of the modified Runge-Lenz vector, $A_i = \frac{1}{(2h)^{1/2}} \left[\frac{1}{2} \left(L_{ij} p_j + p_j L_{ij} \right) - \gamma \frac{\zeta_i}{\sqrt{\zeta^2}} \right]$.

The generators (13-14**)** act in the eigenspace of h equipped with the scalar product

$$(\phi_1, \phi_2) = \int_{R^6} \phi_1^* (\zeta) \, \phi_2 (\zeta) \, d\mu (\zeta) , \quad \zeta \in R^{3n} \tag{17}$$

where $d\mu (\zeta) = \lambda^{-1} (\zeta) \, d\zeta_1 d\zeta_2 \cdots d\zeta_{3n}$.

This representation is, of course, unitarily equivalent to the representation constructed in the eigenspace of the Coulomb Hamiltonian H^{Coul} in $3n$ dimensions.

The operators (13-14) provide most degenerate representations of $SO(3n+1)$ if h is negative definite and of $SO\,(3n,1)$ if h is positive definite. More precisely, they define the most degenerate UIR's of $SO\,(3n+1)$ specified by the integer number $j = 0, 1, \ldots$ when h is negative definite and the most degenerate principal series representations of $SO\,(3n,1)$ labelled by the complex number $j = -\dfrac{3n-1}{2} + i\rho,\ \rho > 0$ when h is positive definite. The Hamiltonian (16) can be related to the operator

$$\begin{aligned} Q \;\; = \;\; & \frac{\gamma^2}{\left[C + \left(\frac{3n-1}{2} \right)^2 \right]} = \frac{\partial^2}{\partial \zeta_1^2} + \frac{\partial^2}{\partial \zeta_2^2} + \ldots + \frac{\partial^2}{\partial \zeta_n^2} \\ & - \frac{2}{\zeta_1^2 + \zeta_2^2 + \zeta_3^2} \left(\zeta_1 \frac{\partial}{\partial \zeta_1} + \zeta_2 \frac{\partial}{\partial \zeta_2} + \zeta_3 \frac{\partial}{\partial \zeta_3} \right) \\ & - \frac{2}{\zeta_4^2 + \zeta_5^2 + \zeta_6^2} \left(\zeta_4 \frac{\partial}{\partial \zeta_4} + \zeta_5 \frac{\partial}{\partial \zeta_5} + \zeta_6 \frac{\partial}{\partial \zeta_6} \right) - \\ & \cdots + \frac{2\gamma}{\sqrt{\zeta^2}} \end{aligned} \tag{18}$$

Here

$$\zeta = (x_1 e_1, x_2 e_2, \ldots, x_n e_n) \tag{19}$$

with $e_i = (\sin \alpha_i \sin \beta_i, \sin \alpha_i \cos \beta_i, \cos \alpha_i)\,,\ i = 1, 2, \ldots, n$, while

$$\begin{aligned} & x_1 = r \sin \theta_{n-1} \sin \theta_{n-2} \ldots \sin \theta_2 \sin \theta_1 \\ & x_2 = r \sin \theta_{n-1} \sin \theta_{n-2} \ldots \sin \theta_2 \cos \theta_1 \\ & \ldots \\ & x_{n-1} = r \sin \theta_{n-1} \cos \theta_{n-2} \\ & x_n = r \cos \theta_{n-1} \end{aligned} \tag{20}$$

Here, the ranges of the polyspherical coordinates (see Ch. 10 of Ref. [4]) are $r \geq 0,\ 0 \leq \theta_i \leq \dfrac{\pi}{2},\ 0 \leq \alpha_k \leq \pi,\ 0 \leq \beta_k < 2\pi$, for $i = 1, 2, \ldots n-1$ and $k =$

$1,2,\ldots n$. Due this parametrization, the operator $Q=\gamma^2\Big/\Big[C+\Big(\dfrac{3n-1}{2}\Big)^2\Big]$ becomes

$$\begin{aligned} Q &= \frac{1}{r^{n-1}}\frac{\partial}{\partial r}r^{n-1}\frac{\partial}{\partial r}+\frac{1}{r^2}\left(\frac{1}{\sin^{n-2}\theta_{n-1}}\frac{\partial}{\partial\theta_{n-1}}\sin^{n-2}\theta_{n-1}\frac{\partial}{\partial\theta_{n-1}}+\ldots\right.\\ &\left.+\frac{1}{\sin^2\theta_{n-1}\sin^2\theta_{n-2}\ldots\sin^2\theta_2}\frac{\partial^2}{\partial\theta_1^2}\right)-\sum_{i=1}^{n}\frac{1}{x_i^2}C^{SO(3)_i}+\frac{2\gamma}{r} \end{aligned} \tag{21}$$

where

$$C^{SO(3)_i}=-\left(\frac{1}{\sin\alpha_i}\frac{\partial}{\partial\alpha_i}\sin\alpha_i\frac{\partial}{\partial\alpha_i}+\frac{1}{\sin^2\alpha_i}\frac{\partial^2}{\partial\beta_i^2}\right)$$

According to this the basis functions $|j;lM\rangle$ can be defined as the common set of eigenfunctions of the Casimir operators of the groups forming the chain

$$\begin{aligned} \underset{j}{G}\supset \underset{m_0}{SO(3n)}\supset \underset{m_1}{SO(3n-3)}\times\underset{\kappa_1}{SO(3)}\supset \underset{m_2}{SO(3n-6)}\times\underset{\kappa_2}{SO(3)}\times SO(3)\supset\\ \ldots\supset \underset{\kappa_n}{SO(3)}\times\underset{\kappa_{n-1}}{SO(3)}\times\ldots\times SO(3) \end{aligned} \tag{22}$$

$$C^G|j;lM\rangle=j(j+3n-1)|j;lM\rangle$$
$$C^{SO(3n-3\nu)}|j;lM\rangle=m_\nu(m_\nu+3n-3\nu-2)|j;lM\rangle\,,\quad \nu=0,1,\ldots,n-2$$
$$C^{SO(3)_i}|j;lM\rangle=\kappa_i(\kappa_i+1)|j;lM\rangle\,,\quad i=1,2,\ldots,n$$

where $m_0\equiv l$, M is the collective index $(m_{1,\cdots,}m_{n-2},\kappa_1,\kappa_2,\ldots,\kappa_n)$ and

$$C^{SO(k)}=\frac{1}{2}\sum_{i,j=1}^{k}M_{ij}^2$$

Let $\mathcal{H}_K$, $K=(\kappa_1,\kappa_2,\ldots,\kappa_n)$, be a subspace spanned by $|j;lM\rangle$ with fixed $\kappa_1,\kappa_2,\ldots,\kappa_n$. Then, the operator (19) restricted to this subspace becomes

$$\left.\frac{\gamma^2}{\Big[C+\Big(\dfrac{3n-1}{2}\Big)^2\Big]}\right|_{\mathcal{H}_K}=\nabla^2-\sum_{i=1}^{n}\frac{\kappa_i(\kappa_i+1)}{x_i^2}+\frac{2\gamma}{r} \tag{23}$$

Hence, the Hamiltonian

$$H=-\frac{1}{2}\nabla^2-\frac{\gamma}{r}+\sum_{i=1}^{n}\frac{\kappa_i(\kappa_i+1)}{2x_i^2}$$

can be described in terms of the potential groups $SO(3n+1)$ and $SO(3n,1)$ since

$$H=-\left.\frac{\gamma^2}{2\Big[C+\Big(\dfrac{3n-1}{2}\Big)^2\Big]}\right|_{\mathcal{H}_K},$$

2.1 Bound States

The bound-state spectrum is immediately obtained from the eigenvalue of the Casimir operator C of the potential group $SO\,(3n+1)$, *i.e.* $j\,(j+3n-1)$, in the form

$$E = -\frac{\gamma^2}{2\left(j+\frac{3n-1}{2}\right)^2}\ , \tag{24}$$

where j takes on integer values from $\kappa_1+\kappa_2+\ldots+\kappa_n$ upwards.

We give for reference the expression of the bound-state wave functions

$$\psi\,(x) = \mathcal{R}_{jl}\,(r)\,\mathcal{Y}_{lM}\,(\hat{x})\,, \tag{25}$$

where $\mathcal{R}_{jl}\,(r)$ is the radial part of the wave function, while $\mathcal{Y}_{lM}\,(\hat{x})$ is the angular part of it :

$$\mathcal{R}_{jl}\,(r) = cu^{l+n}e^{-\frac{u}{2}}L_{j-l}^{2l+3n-2}\,(u)\ \ ,\ u = 4\gamma r/\,(2j+3n-1) \tag{26}$$

with

$$c = (2\gamma)^{-n/2}\left[j+\frac{1}{2}\,(3n-1)\right]^{-\frac{1}{2}(n+1)}\left[\frac{\Gamma\,(j-l+1)}{2\Gamma\,(j+l+3n-1)}\right]^{\frac{1}{2}} \tag{27}$$

and

$$\begin{aligned}\mathcal{Y}_{lM}(\hat{x}) = \chi\prod_{i=1}^{n-2}\sin^{m_i+n-i}\theta_{n-i}\cos^{\kappa_i+1}\theta_{n-i}P_{(m_{i-1}-m_i-\kappa_i)/2}^{\left(m_i+\frac{3n-3i}{2}-1,\,\kappa_i+\frac{1}{2}\right)}(\cos 2\theta_{n-i})&\\ \times\sin^{\kappa_n+1}\theta_1\cos^{\kappa_{n-1}+1}\theta_1 P_{(m_{n-2}-\kappa_n-\kappa_{n-1})/2}^{\left(\kappa_n+\frac{1}{2},\quad\kappa_{n-1}+\frac{1}{2}\right)}(\cos 2\theta_1)&\end{aligned} \tag{28}$$

The normalization constant, χ, is

$$\begin{aligned}\prod_{i=1}^{n-2}\left[\frac{\Gamma\left(\frac{1}{2}\,(m_{i-1}+m_i+\kappa_i+3n-3i+1)\right)\Gamma\left(\frac{1}{2}\,(m_{i-1}-m_i-\kappa_i+2)\right)(2m_i+3n-3i+1)}{\Gamma\left(\frac{1}{2}\,(m_{i-1}+m_i-\kappa_i)\right)\Gamma\left(\frac{1}{2}\,(m_{i-1}-m_i+\kappa_i+3)\right)}\right]^{\frac{1}{2}}&\\ \times\left[\frac{\Gamma\left(\frac{1}{2}\,(m_{n-2}+\kappa_n+\kappa_{n-1}+4)\right)\Gamma\left(\frac{1}{2}\,(m_{n-2}-\kappa_n-\kappa_{n-1}+2)\right)(2m_{n-2}+4)}{\Gamma\left(\frac{1}{2}\,(m_{n-2}+\kappa_n-\kappa_{n-1}+3)\right)\Gamma\left(\frac{1}{2}\,(m_{n-2}-\kappa_n+\kappa_{n-1}+3)\right)}\right]^{\frac{1}{2}}&\end{aligned} \tag{29}$$

Here, L_n^{α} and $P_n^{(\alpha,\beta)}$ are Laguerre and Jacobi polynomials, respectively. It is worth noting that the $\mathcal{Y}_{lM}\,(\hat{x})$ functions are related to $(3n-1)$-dimensional spherical harmonics $Y_{lM}\left(\hat{\zeta}\right)$ (see Section 10.5 of [4]) in polyspherical coordinates as

$$Y_{lM}\left(\hat{\zeta}\right) = \mathcal{Y}_{lM}\,(\hat{x})\prod_{k=1}^{n-1}\left(\sin^{n-k}\theta_{n-k}\cos\theta_{n-k}\right)^{-1}\prod_{i=1}^{n}Y_{\kappa_i}^{\nu_i}\,(\alpha_i,\beta_i)\,. \tag{30}$$

where Y_κ^ν are 2-dimensional spherical harmonics of degree κ, while $\mathcal{R}_{jl}$ is related to the radial part $\mathcal{R}_{jl}^{Coul}(r)$ of the $3n$-dimensional Coulomb wave function [5]

$$\mathcal{R}_{jl}(r) = r^n \mathcal{R}_{jl}^{Coul}(r)$$

Finally, we note that the operators

$$\begin{aligned} I_\nu \quad &= \quad -\frac{1}{2}\sum_{\imath,j=1}^{n-\nu}\left(x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i}\right)^2 + \sum_{l=1}^{n-\nu} x_l^2 \times \sum_{i=1}^{n-\nu}\frac{\kappa_i(\kappa_i+1)}{x_i^2}, \\ & (\nu = 0, 1, \ldots, n-2) \end{aligned}$$

are responsible for the separability of H in spherical coordinates. These integrals of motion are related to the Casimir operators of $SO(3n-3\nu)$ in the sense that

$$I_\nu = C^{SO(3n-3\nu)}\Big|_{\mathcal{H}_K}, \quad (\nu = 0, 1, \ldots, n-2)$$

2.2 Scattering States

Once the group structure of the problem has been recognized, the associated S matrix can be computed by using matrices that intertwine Weyl-equivalent representations of $SO(3n,1)$ in the bases corresponding to the reduction (22). We find it expedient to use, for this purpose, equation (4). By realizing the principal series of $SO(3n,1)$ on suitable Hilbert spaces of appropriate functions, one can derive from (4) the functional relations satisfied by the kernel of the intertwining operator, written in integral form and, consequently, the explicit representation of the matrix elements of the operator itself.

It is known that the most degenerate principal series representations of $SO(N,1)$ labelled with the quantum number $j = -\dfrac{N-1}{2} + i\rho$, with $\rho > 0$, can be realized on $\mathcal{L}_2\left(S^{N-1}\right)$ (see Section 9.2.1 of [4])

$$U_j(g) f(\eta) = (\omega_g)^j f(\eta_g), \quad \eta \in S^{N-1} \tag{31}$$

where

$$\omega_g = \sum_{i=1}^{N} g_{Ni}^{-1}\eta_i + g_{NN}, \quad (\eta_g)_k = \frac{\sum_{i=1}^N g_{ki}^{-1}\eta_i + g_{kN}}{\sum_{i=1}^N g_{Ni}^{-1}\eta_i + g_{NN}}$$

The representations specified by labels j and $1-N-j$ are Weyl-equivalent.

The operator A defined by

$$(Af)(\eta) = \int K(\eta,\eta') f(\eta') d\eta' \tag{32}$$

intertwines representations j and $1-N-j$ on condition that

$$K\left(\eta_g, \eta'_g\right) = (\omega_g)^{N-1+j}\left(\omega'_g\right)^{N-1+j} K(\eta,\eta'). \tag{33}$$

The kernel, K , is uniquely determined by Eq. (33) up to a constant and is given by

$$K(\eta,\eta') = \varkappa\left(1-\eta\cdot\eta'\right)^{1-N-j} \ . \tag{34}$$

with

$$\varkappa = 2^{-\frac{N-1}{2}+i\rho}\frac{\Gamma\left(\frac{N-1}{2}+i\rho\right)}{\pi^{\frac{N-1}{2}}\Gamma\left(-i\rho\right)} \tag{35}$$

Taking into account the fact that the $(N-1)$-dimensional spherical harmonics Y_{lM} of degree l [4] forms a basis in $\mathcal{L}_2\left(S^{N-1}\right)$, corresponding to the above reduction, we obtain the following integral representation for the matrix elements of A

$$\langle j; l'M'|\, A\,|j; lM\rangle = \int K(\eta,\eta')Y^*_{l'M'}\left(\eta'\right)Y_{lM}\left(\eta\right)d\eta d\eta' \ . \tag{36}$$

Therefore

$$\langle j; l'M'|\, A\,|j; lM\rangle = A_l\delta_{ll'}\delta_{MM'} \ , \tag{37}$$

where

$$A_l = \frac{\Gamma\left(\frac{N-1}{2}+i\rho+l\right)}{\Gamma\left(\frac{N-1}{2}-i\rho+l\right)}. \tag{38}$$

According to this, we have

$$S\left(p;p'\right) = \sum_{lM} A_l\mathcal{Y}_{lM}\left(\hat{p}\right)\mathcal{Y}^*_{lM}\left(\hat{p}'\right) \ . \tag{39}$$

Thus, the scattering amplitude, $f\left(p;p'\right)$, is defined by

$$f\left(p;p'\right) = (-i)\left(\frac{2\pi}{p}\right)^{\frac{n-1}{2}}\sum_{lM}\left(A_l-1\right)\mathcal{Y}_{lM}\left(\hat{p}\right)\mathcal{Y}^*_{lM}\left(\hat{p}'\right) \ . \tag{40}$$

We can omit unity in the brackets of formula (40) when $\hat{p}'\neq\hat{p}$, leaving

$$f\left(p;p'\right) = (-i)\left(\frac{2\pi}{p}\right)^{\frac{n-1}{2}}\sum_{lM}\frac{\Gamma\left(\frac{3n-1}{2}+i\rho+l\right)}{\Gamma\left(\frac{3n-1}{2}-i\rho+l\right)}\mathcal{Y}_{lM}\left(\hat{p}\right)\mathcal{Y}^*_{lM}\left(\hat{p}'\right) \ . \tag{41}$$

Moreover, formulas (30) and the following expansion of the kernel

$$\begin{aligned}\left(1-\eta\cdot\eta'\right)^{-\frac{3n-1}{2}-i\rho} &= (2\pi)^{\frac{3n-1}{2}}\ \frac{2^{-i\rho}\Gamma\left(-i\rho\right)}{\Gamma\left(\frac{3n-1}{2}+i\rho\right)} \\ &\times \sum_{lM}\frac{\Gamma\left(\frac{3n-1}{2}+i\rho+l\right)}{\Gamma\left(\frac{3n-1}{2}-i\rho+l\right)}Y^*_{l'M'}\left(\eta'\right)Y_{lM}\left(\eta\right) \ ,\end{aligned}$$

yield an integral representation of the scattering amplitude

$$\begin{aligned} f\left(p;p'\right) &= \frac{1}{ip^{\frac{n-1}{2}}}\frac{2^{i\rho}\Gamma\left(\frac{3n-1}{2}+i\rho\right)}{\Gamma\left(-i\rho\right)}\Lambda\left(\theta_{n-1},\ldots,\theta_1\right)\Lambda\left(\theta'_{n-1},\ldots,\theta'_1\right) \\ &\times\int_0^\pi\cdots\int_0^\pi\left(1-\sum_{j=1}^n\hat{p}_j\hat{p}'_j\cos\alpha_j\right)^{-\frac{3n-1}{2}-i\rho} \\ &\times\prod_{l=1}^n P_{\kappa_l}\left(\cos\alpha_l\right)\sin\alpha_l d\alpha_l \end{aligned} \tag{42}$$

where

$$\Lambda\left(\theta_{n-1},\ldots,\theta_1\right)=\prod_{i=1}^{n-1}\sin^{n-i}\theta_{n-i}\cos\theta_{n-i}$$

When $\kappa_i=0$, formula (42) simplifies to

$$f\left(p;p'\right)=\frac{1}{ip^{\frac{n-1}{2}}}\frac{2^{i\rho}\Gamma\left(\frac{n-1}{2}+i\rho\right)}{\Gamma\left(-i\rho\right)}\sum_{\epsilon_i=\pm}\sigma\left(1-\sum_{j=1}^n\epsilon_j\hat{p}_j\hat{p}'_j\right)^{-\frac{n-1}{2}-i\rho} \tag{43}$$

with

$$\sigma=\prod_{i=1}^n\epsilon_i$$

It is worth noting that that the amplitude (43) does not reduce to the Coulomb amplitude in n dimensions [6]

$$f_{Coul}(p;p')=\frac{1}{ip^{\frac{n-1}{2}}}\frac{2^{i\rho}\Gamma\left(\frac{n-1}{2}+i\rho\right)}{\Gamma(-i\rho)}(1-\hat{p}_1\hat{p}'_1-\hat{p}_2\hat{p}'_2-\cdots-\hat{p}_n\hat{p}'_n)^{-\frac{n-1}{2}-i\rho} \tag{44}$$

when κ_i is set equal to zero. The reason for this discrepancy lies in the fact that the Schrödinger equation with potential (1) is supplemented with the following boundary condition on the wave function at $x_i=0$, $(i=0,1,\ldots,n)$, where the Coulomb-Rosochatius potential is singular

$$\Psi\left(x\right)=0 \quad \text{if} \quad x_i=0 \quad \left(i=0,1,\ldots,n\right)$$

3 Conclusions and Outlook

The present work is the latest in a series of papers where the potential group approach and the method of intertwining operators (formulae (2-7)) have been applied to non-central extensions of the Coulomb potential: Ref. [7] studied the bound states of the three-dimensional Coulomb potential plus a barrier term with the $SO(5)$ potential group and Ref. [8] the scattering states of similar potentials with $SO(5,1)$ potential group. A first simultaneous analysis of bound and scattering states of a three-dimensional Coulomb-Rosochatius potential of type (1) was performed in Ref. [9], where the use of potential groups $SO(7)$ for bound states and $SO(6,1)$ for scattering states permitted the algebraic solution of a sub-family of potentials (1) not including the pure Coulomb potential, since the potential strengths β_i ($i = 1, 2, 3$) allowed by the final subgroup in the reduction chain, $SO(2) \times SO(2) \times SO(2)$, cannot be set to zero. In the present work, the choice of potential groups $SO(3n+1)$ and $SO(3n,1)$ in the n-dimensional case yields reduction chains with final subgroups that are the direct product of n copies of $SO(3)$, so that the allowed potential strengths now are non-negative integers, thus including the pure n-dimensional Coulomb potential as a particular case ($\beta_i = 0$, $i = 1, ..., n$). The scattering amplitude computed for $\beta_i = 0$, however, does not coincide with the Coulomb scattering amplitude in n dimensions, owing to the different boundary conditions imposed to the wave function in the Schrődinger equation.

The potential group approach is quite general and, obviously, not limited to the orthogonal and pseudo-orthogonal symmetries underlying the Coulomb-Rosochatius Hamiltonian. Systems with different symmetries will be studied in future works.

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