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SUSYQM in Nuclear Structure: Bohr Hamiltonian with Deformation-Dependent Mass Term

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Abstract.

The Bohr Hamiltonian describing the collective motion of atomic nuclei was modified recently in [1] by allowing the mass to depend on the nuclear deformation. In this article an extensive review of the above procedure is presented with the aim to provide signs for its generalisation to the mass tensor case.

1 Introduction

The behavior of mass in all the range of nuclear structure is a fundamental problem of nuclear physics. From the liquid drop to the interacting bosons decades passed in which the problem of mass was faced mainly by microscopic methods.

The Bohr Hamiltonian [2] and its extension, the geometric collective model (GCM) [3], revealed the strength of the liquid drop picture, which is the understanding of the nuclear collectivity, mainly through the discrimination of spherical and deformed nuclei. The emphasis in the particle structure was due to the weakness of the GCM to agree with the experimental data for the masses of deformed nuclei. A characteristic feature of this weakness views the behavior of the moments of inertia for deformed nuclei. They are predicted to increase proportionally to β^2 [4], where β is the collective variable corresponding to the axial deformation, while experimentally a much more moderate increase is observed.

The microscopic methods for the description of the collective features of atomic nuclei were introduced by Bohr-Mottelson-Pines [5], and independently by Bogolyubov [6], with the rising of the nuclear pair correlations. The comprehension of nuclear superfluidity gave the beautiful explanation for the mass difference between even-even and odd nuclei. Furthermore, in the spherical nuclei the energy gap appearing between the ground and the first excited state, is significantly larger than that for the deformed nuclei [7] and an effective quasi-particle mass was introduced by Migdal [8], in order to make a more experimentaly consistent prediction for the nuclear moments of inertia of deformed nuclei. The spontaneous breakdown of spherical symmetry [9] during the transition from a superfluid nucleus to a deformed one, signaled the entrance of mean field theories such as Hartree-Fock-Bogolyubov theories and the Random Phase Approximation (RPA). The main result of these efforts, regarding the behavior of mass in nuclear collectivity, [10] is that the mass coefficients should be different for the vibrational and the rotational degrees of freedom. Recently an argumentation is being developed [11], based on experimental data for various nuclei, which amounts to consider the mass coefficient of the Bohr Hamiltonian, not as a constant scalar quantity but as a mass tensor with variable elements obtained from experimental data. Therefore the microscopic results for the mass behavior in nuclear structure can be hosted in the liquid drop picture, by first considering a non constant mass, and afterwards generalizing it to the tensor case.

On the other hand, the symmetries of the liquid drop can in principle be obtained from a second quantization procedure, and this is the algebraic framework of the Interacting Boson Model (IBM) [12], which classifies the various types of nuclear collective motion with distinguished simplicity. The states of spherical nuclei "sit" in the representations of U(5) symmetry, the states of γ-unstable nuclei "sit" in the representations of O(6) symmetry and those of the axial deformed nuclei "sit" in the representations of SU(3) symmetry. The SU(3) symmetry follows the experimental data and reveals the importance of IBM, as such a symmetry has not been obtained yet from the Bohr Hamiltonian.

As the second quantization is always based on the canonical commutation relations, it can be stated that the algebraic methods of the IBM can give an insight for the canonical quantization of the liquid drop and more directly, the canonical quantization of the liquid drop must correspond to the classical limit of the IBM. Concerning mass, in the classical limit of the IBM, in addition to the usual term of the kinetic energy, π^2, terms of the form $\beta^2\pi^2$ appear in the O(6) limiting symmetry and in the U(5)-O(6) transition region, while more complicated terms appear in the SU(3) limiting symmetry, as well as in the U(5)-SU(3) and SU(3)-O(6) transition regions [13]. These terms clearly challenge us to consider deformation effects on the momentum operator, that is the momentum operator should get dressed with terms of deformation. A mathematicaly self-consistent way to obtain such a type of quantization is the deformed algebras. Quesne and Tkachuk [14] have demonstrated that, under certain circumstances, deformed canonical commutation relations are equivalent to a position dependent mass and also to a curved space.

Our analysis is based on the above two axes. The first is the modification of the Bohr Hamiltonian with a non constant mass coefficient of the kinetic term, while the second is the classical limit of the IBM which indicates the deformation of the canonical commutation relations of the liquid drop. Taking into advantage the "Quesne and Tkachuk equivalence" and the presence of β in the classical limit of the IBM for the momentum operator, the Bohr Hamiltonian is modified with a mass depending on the collective variable β. The deformation of the canonical commutation relations is not necessary at this level, although it will be proved to be the path to the tensor mass generalization.

2 The Quesne and Tkachuk Equivalence and Position Dependent Mass Bohr Hamiltonian

A mathematicaly self-consistent way to obtain deformed operators is that of the deformed algebras, where the usual canonical quantization

$$[\mathbf{x}, \mathbf{p}] = i\hbar, \tag{1}$$

is modified by the presence of a so called deformation function, which in principle may depend on position, momentum, or their combinations. Quesne and Tkachuk [14] showed that if this deformation is a function of the position, that is

$$[\mathbf{x}, \mathbf{p}] = i\hbar f(\mathbf{x}), \tag{2}$$

then the following equivalence holds (the Quesne and Tkachuk equivalence)

$$f^2(\mathbf{x}) = \frac{1}{M(\mathbf{x})} = \frac{1}{g(\mathbf{x})}, \tag{3}$$

where $1/M(\mathbf{x})$ is an inverted position dependent mass and $1/g(\mathbf{x})$ is an inverted diagonalized metric. In this case, the momentum operator is deformed as $\sqrt{f(\mathbf{x})}\mathbf{p}\sqrt{f(\mathbf{x})}$. In order to follow the results of the IBM for the β-dependent momentum and the path to a non-constant mass coefficient in the Bohr Hamiltonian, in the beginning we consider only the first part of this equivalence. Therefore we review [14] now the general effects of a quantum mechanical system with a position dependent mass.

A Position Dependent Mass (PDM) $m(\mathbf{x})$ [14], does not commute with the momentum $\mathbf{p} = -i\hbar\nabla$. As a consequence, there are many ways to generalize the usual form of the kinetic energy, $\mathbf{p}^2/(2m_0)$, where m_0 is a constant mass, in order to obtain a Hermitian operator. In order to avoid any specific choices, one can use the general two-parameter form proposed by von Roos [15], with a Hamiltonian

$$H = -\frac{\hbar^2}{4}[m^{\delta'}(\mathbf{x})\nabla m^{\kappa'}(\mathbf{x})\nabla m^{\lambda'}(\mathbf{x}) + m^{\lambda'}(\mathbf{x})\nabla m^{\kappa'}(\mathbf{x})\nabla m^{\delta'}(\mathbf{x})] + V(\mathbf{x}), \tag{4}$$

where V is the relevant potential and the parameters δ', κ', λ' are constrained by the condition $\delta' + \kappa' + \lambda' = -1$. Assuming a position dependent mass of the form

$$m(\mathbf{x}) = m_0 M(\mathbf{x}), \qquad M(\mathbf{x}) = \frac{1}{(f(\mathbf{x}))^2}, \qquad f(\mathbf{x}) = 1 + g(\mathbf{x}), \tag{5}$$

where m_0 is a constant mass and $M(\mathbf{x})$ is a dimensionless position-dependent mass, we obtain the Hamiltonian

$$H = -\frac{\hbar^2}{2m_0}\sqrt{f(\mathbf{x})}\nabla f(\mathbf{x})\nabla\sqrt{f(\mathbf{x})} + V_{eff}(\mathbf{x}), \tag{6}$$

with

$$V_{eff}(\mathbf{x}) = V(\mathbf{x}) + \frac{\hbar^2}{2m_0}\left[\frac{1}{2}(1-\delta-\lambda)f(\mathbf{x})\nabla^2 f(\mathbf{x}) + \left(\frac{1}{2}-\delta\right)\left(\frac{1}{2}-\lambda\right)(\nabla f(\mathbf{x}))^2\right]. \tag{7}$$

Finaly, a PDM Hamiltonian is recognized by the following two features, the modification of the Laplacian operator by the form $\sqrt{f(\mathbf{x})}\nabla f(\mathbf{x})\nabla\sqrt{f(\mathbf{x})}$, and the presence of an effective potential which emerges from higher orders of the gradient of this deformation function.

What was done in [1]was the application of the PDM framework in the case of the Bohr Hamiltonian, for a mass dependent on the β variable. Therefore first a mass dependence of the form

$$B(\beta) = \frac{B_0}{(f(\beta))^2}, \tag{8}$$

where B_0 is a constant, was assumed. As the dependence is a scalar one, it permits us to follow the usual Pauli–Podolsky prescription. Since the deformation function f depends only on the radial coordinate β, it is expected that the modification of the Laplacian operator will affect the β part of the resulting equation. Therefore, according to the general features of a PDM Hamiltonian, we expect to obtain an effective potential and a modified derivative for the β part of the Bohr Hamiltonian, the final result reading

$$\left[-\frac{1}{2}\frac{\sqrt{f}}{\beta^4}\frac{\partial}{\partial\beta}\beta^4 f\frac{\partial}{\partial\beta}\sqrt{f} - \frac{f^2}{2\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} + \frac{f^2}{8\beta^2}\sum_{k=1,2,3}\frac{Q_k^2}{\sin^2\left(\gamma-\frac{2}{3}\pi k\right)} + v_{eff}\right]\Psi = \epsilon\Psi, \tag{9}$$

where reduced energies $\epsilon = B_0E/\hbar^2$ and reduced potentials $v = B_0V/\hbar^2$ have been used. The angular part (γ, θ_i) is affected in a different way, only by the presence of the coefficient $f^2/(8\beta^2)$, which will be shown to be crucial for the reduction of the moments of inertia. The other feature, that of the effective potential, takes the following form in the case of a PDM-Bohr Hamiltonian

$$v_{eff} = v(\beta,\gamma) + \frac{1}{4}(1-\delta-\lambda)f\nabla^2 f + \frac{1}{2}\left(\frac{1}{2}-\delta\right)\left(\frac{1}{2}-\lambda\right)(\nabla f)^2. \tag{10}$$

3 The Deformed Radial Equation for the γ-unstable Davidson Potential

The solution of the above Bohr-like equation can be reached for certain classes of potentials using techniques developed in the context of supersymmetric quantum mechanics (SUSYQM) [16, 17]. Furthermore the integrability condition of the Shape Invariance which is of great use in the above mentioned techniques, is the key for determining the deformation function, which in principle is unknown. As a first test of the PDM Bohr Hamiltonian we applied it to the case of γ-unstable nuclei, as they are very close to spherical nuclei and depart from axial symmetry without any energy cost. If an almost spherical nucleus displays position dependent mass features, then a PDM framework will certainly be necessary for the nuclear matter that is organized in a deformed shape.

In order to achieve separation of variables we assumed that the potential $v(\beta,\gamma)$ depends only on the variable β, i.e. $v(\beta,\gamma) = u(\beta)$ [18]. For the radial potential we used the Davidson potential [19]

$$u(\beta) = \beta^2 + \frac{\beta_0^4}{\beta^2}. \tag{11}$$

The separation of variables is presented in [1]. Here we state the result for the radial part. With

$$v_{eff} = u + \frac{1}{4}(1-\delta-\lambda)f\left(\frac{4f'}{\beta} + f''\right) + \frac{1}{2}\left(\frac{1}{2}-\delta\right)\left(\frac{1}{2}-\lambda\right)(f')^2, \tag{12}$$

the radial equation takes its deformed version

$$HR = -\frac{1}{2}\left(\sqrt{f}\frac{d}{d\beta}\sqrt{f}\right)^2 R + u_{eff}R = \epsilon R, \tag{13}$$

where

$$u_{eff} = v_{eff} + \frac{f^2 + \beta f f'}{\beta^2} + \frac{f^2}{2\beta^2}\Lambda. \tag{14}$$

4 SUSY QM, Shape Invariance and the Deformation Function

Following the general method used in SUSYQM [16], one should take the following steps:

i) Factorize the Hamiltonian in terms of generalized ladder operators

$$H_0 = B_0^+ B_0^- + \varepsilon_0. \tag{15}$$

ii) Write a hierarchy of Hamiltonians starting from the first one. This means that the above Hamiltonian should be considered as the first part of the series

$$H_i = B_i^+ B_i^- + \sum_{j=0}^{i} \varepsilon_j. \tag{16}$$

The determination of these operators requires the conceptual tool of the shape invariance for the following reasons. First, the deformation function that modifies the momentum operator is not yet determined. Second, the superpotential of the Davidson potential is a known one [16,17], yet the superpotential that we are looking for should correspond to the effective potential which is also unknown, because of the presence of the deformation function.

iii) Impose the shape invariance conditions, which are integrability conditions guaranteeing exact solvability.

This condition is the crucial one for the determination of the deformation function. The effective potential should be shape invariant, which means that it has to retain the same functional dependence in the whole hierarchy of Hamiltonians. If the mass is not position dependent, the effective potential is absent. This may be the situation for a transitional nucleus, which is between the spherical and γ-unstable behavior. Such a nucleus has a very small, yet finite value for β_0 and therefore the Davidson behavior is appropriate. Now, if there are slight deviations of this behavior with respect to extremely tiny changes of the β_0, these deviations should be attributed to an effective mass and this can be realized in part by the effective potential of the PDM framework. The shape invariance condition states that all these tiny changes should be generated only by a “fine tuning” of some parameters, leaving the functional dependence of the Davidson potential shape invariant.

The deformation function is unknown. Its definition should bring a Davidson behavior to the effective potential, as the last is guided by the shape invariance. Thus in the last equation of Section 3, we want to have only the terms of the Davidson potential, that is terms proportional to the β^2 and proportional to the $1/\beta^2$. The choice for the deformation function

$$f(\beta) = 1 + a\beta^2, \tag{17}$$

gives the Davidson behavior to the effective potential,

$$\begin{aligned} u_{eff} = \beta^2 + a^2\beta^2 \left[\frac{5}{2}(1-\delta-\lambda) + 2\left(\frac{1}{2}-\delta\right)\left(\frac{1}{2}-\lambda\right) + 3 + \frac{\Lambda}{2}\right] \\ + \frac{1}{\beta^2}\left(1 + \frac{1}{2}\Lambda + \beta_0^4\right) + a\left[\frac{5}{2}(1-\delta-\lambda) + 4 + \Lambda\right]. \end{aligned} \tag{18}$$

The parameter a is called the deformation parameter. The mass is position dependent for non-zero values of a. Therefore the values of β_0 and a show first the magnitude of the deformation and second the position dependence of the mass. For tiny changes of β_0, the parameter a varies in the Davidson behavior and this variation reflects the position dependence of the mass during the transition. In Section 5 this situation is clarified.

Here it is worth pointing out the effect of the deformation function in the coefficient of the angular part of the PDM Bohr Hamiltonian. From Eq. (9) it is clear that in the present case the moments of inertia are not proportional to

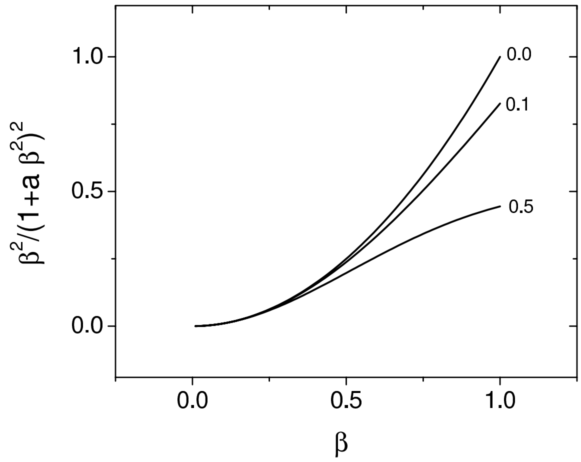

Figure 1. The function $\beta^2/f^2(\beta) = \beta^2/(1+a\beta^2)^2$, to which moments of inertia are proportional as seen from Eq. (9), plotted as a function of the nuclear deformation β for different values of the parameter a. See Secttion 4 for further discussion.

$\beta^2 \sin^2(\gamma - 2\pi k/3)$ but to $(\beta^2/f^2(\beta))\sin^2(\gamma - 2\pi k/3)$.The function $\beta^2/f^2(\beta)$ is shown in Figure 1 for different values of the parameter a. It is clear that the increase of the moment of inertia is slowed down by the function $f(\beta)$, as it is expected as nuclear deformation sets in [4].

We state again that the shape invariance condition was the tool to determine the deformation function. This condition in terms of the ladder operators reads

$$B_i^- B_i^+ = B_{i+1}^+ B_{i+1}^- + \varepsilon_{i+1}. \tag{19}$$

It is obvious now that the superpotential of the effective potential should be the one corresponding to the Davidson potential, which is known [17]. Therefore the ladder operators of the first member of the hierarchy should take the form

$$B_0^{\pm} = \mp \frac{1}{\sqrt{2}} \left(\sqrt{f} \frac{d}{d\beta} \sqrt{f} \right) + \frac{1}{\sqrt{2}} \left(c_0 \beta + \bar{c}_0 \frac{1}{\beta} \right), \tag{20}$$

and those of the next members of the hierarchy of Hamiltonians should take the form

$$B_i^{\pm} = \mp \frac{1}{\sqrt{2}} \left(\sqrt{f} \frac{d}{d\beta} \sqrt{f} \right) + \frac{1}{\sqrt{2}} \left(c_i \beta + \frac{\bar{c}_i}{\beta} \right). \tag{21}$$

From these equations the eigenvalues and eigenfunctions are obtained in [1]. For example the energies of the ground state band are found to be

$$\varepsilon_0 = 7a \left(\frac{29}{4} - \frac{5}{2}(\delta + \lambda) + \Lambda \right) + \frac{1}{2}\sqrt{a^2 + 8P_1} + \frac{a}{2}\sqrt{9 + 4\Lambda + 8\beta_0^4} + \frac{1}{4}\sqrt{(a^2 + 8P_1)(9 + 4\Lambda + 8\beta_0^4)}. \tag{22}$$

About the values of the von Roos parameters, δ and λ it must be mentioned that although their presence is essential in the kinetic part of the von Roos Hamiltonian, in the PDM framework after the Quesne and Tkachuk equivalence, δ and λ appeared only in the effective potential and not in the kinetic part. With the application of the shape invariance condition in the effective potential, a change in δ and λ automatically renormalizes the parameter values of a and β_0. Therefore for the whole hierarchy we can give a fixed value for δ and λ, as they do not contribute independently to the energy spectrum. So in Section 5 we set their value to be $\delta = \lambda = 0$.

5 Numerical Results

As a first testground of the present method we have used the Xe isotopes shown in Table 1. Their choice is justified in [1]. It is worth considering here the values of the parameters in each nucleus.

i) ^{134}Xe and ^{132}Xe are almost pure vibrators. Therefore no need for deformation dependence of the mass exists, the least square fitting leading to $a = 0$. Furthermore, no β_0 term is needed in the potential, the fitting therefore leading to $\beta_0 = 0$, *i.e.*, to pure harmonic behaviour.

ii) In the next two isotopes (^{130}Xe and ^{128}Xe) the need to depart from the pure harmonic oscillator becomes clear, the fitting leading therefore to nonzero β_0 values. However, there is still no need of dependence of the mass on the deformation, the fitting still leading to $a = 0$. Even if we have a finite value for the β_0, the mass is not yet position dependent. But in ^{126}Xe, for the same

Table 1. Comparison of theoretical predictions of the γ-unstable Bohr Hamiltonian with β-dependent mass (with $\delta = \lambda = 0$) to experimental data [20] of Xe isotopes. The $R_{4/2} = E(4_1^+)/E(2_1^+)$ ratios (indicated as $4/2$ in the table), as well as the quasi-β_1 and quasi-γ_1 bandheads, normalized to the 2_1^+ state and labelled by $R_{0/2} = E(0_\beta^+)/E(2_1^+)$ and $R_{2/2} = E(2_\gamma^+)/E(2_1^+)$ respectively (indicated as $0/2$ and $2/2$ in the table), are shown. n indicates the total number of levels involved in the fit and σ is the quality measure.

	$4/2$ exp	$4/2$ th	$0/2$ exp	$0/2$ th	$2/2$ exp	$2/2$ th	β_0	a	n	σ
^{118}Xe	2.40	2.32	2.5	2.6	2.8	2.3	1.27	0.103	19	0.319
^{120}Xe	2.47	2.36	2.8	3.4	2.7	2.4	1.51	0.063	23	0.524
^{122}Xe	2.50	2.40	3.5	3.3	2.5	2.4	1.57	0.096	16	0.638
^{124}Xe	2.48	2.36	3.6	3.5	2.4	2.4	1.55	0.051	21	0.554
^{126}Xe	2.42	2.33	3.4	3.1	2.3	2.3	1.42	0.064	16	0.584
^{128}Xe	2.33	2.27	3.6	3.5	2.2	2.3	1.42	0.000	12	0.431
^{130}Xe	2.25	2.21	3.3	3.1	2.1	2.2	1.27	0.000	11	0.347
^{132}Xe	2.16	2.00	2.8	2.0	1.9	2.0	0.00	0.000	7	0.467
^{134}Xe	2.04	2.00	1.9	2.0	1.9	2.0	0.00	0.000	7	0.685

value of β_0 as in ^{128}Xe, the mass is position dependent. These three nuclei (^{130}Xe, ^{128}Xe, and ^{126}Xe) seem to be good candidates for the examination of the behavior of the mass during the phase transition from a spherical to a γ-unstable behavior. What is illustrated here is the changes of the effective potential and the PDM during tiny changes of the deformation.

iii) Beyond ^{126}Xe both the β_0 term in the potential and the deformation dependence of the mass become necessary, leading to nonzero values of both β_0 and a.

6 Introduction to the Tensor Generalization

For a Hamiltonian that describes the dynamics of a system in a curved space, the Poisson brackets take the form,

$$[x_\mu, p_\nu] = g_{\mu\nu}, \tag{23}$$

with $g_{\mu\nu}$ being the metric tensor of the underlying space. The quantization of the above phase space can be immediately obtained, as for instance is stated in [21], by the commutation relations

$$[x_\mu, p_\nu] = i\hbar g_{\mu\nu}. \tag{24}$$

In a generalized Riemmannian manifold as can be found in [22], an inverted metric $(\Theta_{\mu\nu})^{-1}$ is equal to the double commutator

$$(\Theta_{\mu\nu})^{-1} = [Q_{2\mu}, [H, Q_{2\nu}]] = -[[H, Q_{2\mu}], Q_{2\nu}]. \tag{25}$$

In [11] it is shown that this double commutator is an inverted mass tensor. In addition, based on experimental data, its existence in the Bohr Hamiltonian is proved. Therefore, the commutation relations for the quadrupole degree of freedom $a_{2\mu}$ and its generalized momenta $\pi_{2\mu}$ should be generalized to the tensor form

$$[\alpha_{2\mu}, \pi_{2\nu}] = i\hbar\Theta_{\mu\nu}. \tag{26}$$

7 Conclusion

Based on the classical limit of the IBM and on the approach of a non constant mass coefficient in the Bohr Hamiltonian, a PDM Bohr Hamiltonian is obtained. Its application to γ-unstable nuclei gives encouranging results. Furthermore, the mass behavior during the transition from the spherical to the γ-unstable case seems to be promising. In general, the mass tensor in the Bohr Hamiltonian signals the deformation of the canonical quantization of the liquid drop and this is in line with the results of the classical limit of the IBM.

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