

# Conformal Maps and Group Contractions in Nuclear Structure

## D. Bonatsos

Institute of Nuclear Physics, N.C.S.R. "Demokritos", GR-15310 Aghia Paraskevi, Attiki, Greece

### **Abstract.**

In mathematics, a conformal map is a function which preserves angles. We show how this procedure can be used in the framework of the Bohr Hamiltonian, leading to a Hamiltonian in a curved space, in which the mass depends on the nuclear deformation  $\beta$ , while it remains independent of the collective variable  $\gamma$  and the three Euler angles. This Hamiltonian is proved to be equivalent to that obtained using techniques of Supersymmetric Quantum Mechanics.

Group contraction is a procedure in which a symmetry group is reduced into a group of lower symmetry in a certain limiting case. Examples are provided in the large boson number limit of the Interacting Boson Approximation (IBA) model by a) the contraction of the SU(3) algebra into the  $[R^5]$ SO(3) algebra of the rigid rotator, consisting of the angular momentum operators forming SO(3), plus 5 mutually commuting quantities, the quadrupole operators, b) the contraction of the O(6) algebra into the  $[R^5]$ SO(5) algebra of the  $\gamma$ -unstable rotator. We show how contractions can be used for constructing symmetry lines in the interior of the symmetry triangle of the IBA model.

## **1 Introduction**

The Interacting Boson Approximation (IBA) model [1] and the collective model of Bohr [2] provide complementary descriptions of the properties of medium-mass and heavy nuclei.

In the present work we first show how the Bohr Hamiltonian can be generalized into a curved space [3]. This generalization turns out to be equivalent to the use of a nuclear mass which is not a constant, as in the usual case, but depends on the nuclear deformation. This modification solves [3] a long-standing problem of nuclear structure, namely the too rapid increase of the moment of inertia with deformation [4].

Furthermore in this work we are going to show [5] how the mathematical concept of group contraction [6] can be used for determining approximate symmetries in the framework of the IBA model [1]. In particular, an approximate SU(3) symmetry will be determined within the symmetry triangle [7] of the IBA model, at the vertices of which the dynamical symmetries of the model appear.

## 2 Conformal maps

### 2.1 A formulation of the Bohr Hamiltonian

The original Bohr Hamiltonian [2] is

$$H_B = -\frac{\hbar^2}{2B} \left[ \frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{1}{\beta^2 \sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} - \frac{1}{4\beta^2} \sum_{k=1,2,3} \frac{Q_k^2}{\sin^2(\gamma - \frac{2}{3}\pi k)} \right] + V(\beta, \gamma), \quad (1)$$

where  $\beta$  and  $\gamma$  are the usual collective coordinates ( $\beta$  being a deformation coordinate measuring departure from spherical shape, and  $\gamma$  being an angle measuring departure from axial symmetry), while  $Q_k$  ( $k = 1, 2, 3$ ) are the components of angular momentum in the intrinsic frame, and  $B$  is the mass parameter, which is usually considered constant.

In Ref. [8] it has been proved that the position-dependent effective mass formalism can be equivalently expressed in a curved space. We shall prove here that this connection is possible also in the case of the Bohr Hamiltonian.

Ordering the coordinates as

$$q_1 = \Phi, \quad q_2 = \Theta, \quad q_3 = \psi, \quad q_4 = \beta, \quad q_5 = \gamma, \quad (2)$$

the kinetic energy in the standard Bohr Hamiltonian [2] can be represented as

$$T = \frac{B}{2} \left( \frac{ds}{dt} \right)^2, \quad (3)$$

where

$$ds^2 = g_{ij} dq_i dq_j, \quad (4)$$

the symmetric matrix  $g_{ij}$  having the form

$$(g_{ij}) = \begin{pmatrix} g_{11} & g_{12} & g_{13} & 0 & 0 \\ g_{21} & g_{22} & 0 & 0 & 0 \\ g_{31} & 0 & g_{33} & 0 & 0 \\ 0 & 0 & 0 & g_{44} & 0 \\ 0 & 0 & 0 & 0 & g_{55} \end{pmatrix}, \quad (5)$$

with

$$g_{11} = \frac{J_1}{B} \sin^2 \Theta \cos^2 \psi + \frac{J_2}{B} \sin^2 \Theta \sin^2 \psi + \frac{J_3}{B} \cos^2 \Theta, \quad (6)$$

$$g_{12} = \frac{1}{B} (J_2 - J_1) \sin \Theta \sin \psi \cos \psi, \quad g_{13} = \frac{J_3}{B} \cos \Theta, \quad (7)$$

$$g_{22} = \frac{J_1}{B} \sin^2 \psi + \frac{J_2}{B} \cos^2 \psi, \quad g_{33} = \frac{J_3}{B}, \quad g_{44} = 1, \quad g_{55} = \beta^2, \quad (8)$$

where the moments of inertia are

$$J_k = 4B\beta^2 \sin^2 \left( \gamma - k \frac{2\pi}{3} \right). \quad (9)$$

The determinant of the matrix is

$$g = \frac{J_1 J_2 J_3}{B^3} \beta^2 \sin^2 \Theta = 4\beta^8 \sin^2 3\gamma \sin^2 \Theta. \quad (10)$$

The relevant volume element is then

$$dV = 2\beta^4 \sin 3\gamma \sin \Theta d\Phi d\Theta d\psi d\beta d\gamma. \quad (11)$$

The inverse matrix is found to be

$$(g_{ij}^{-1}) = \begin{pmatrix} g_{11}^{-1} & g_{12}^{-1} & g_{13}^{-1} & 0 & 0 \\ g_{21}^{-1} & g_{22}^{-1} & g_{23}^{-1} & 0 & 0 \\ g_{31}^{-1} & g_{32}^{-1} & g_{33}^{-1} & 0 & 0 \\ 0 & 0 & 0 & g_{44}^{-1} & 0 \\ 0 & 0 & 0 & 0 & g_{55}^{-1} \end{pmatrix}, \quad (12)$$

with

$$g_{11}^{-1} = \frac{B}{\sin^2 \Theta} \left( \frac{\cos^2 \psi}{J_1} + \frac{\sin^2 \psi}{J_2} \right), \quad (13)$$

$$g_{12}^{-1} = -B \left( \frac{1}{J_1} - \frac{1}{J_2} \right) \frac{\sin \psi \cos \psi}{\sin \Theta}, \quad (14)$$

$$g_{13}^{-1} = -B \left( \frac{\cos^2 \psi}{J_1} + \frac{\sin^2 \psi}{J_2} \right) \frac{\cot \Theta}{\sin \Theta}, \quad g_{22}^{-1} = B \left( \frac{\sin^2 \psi}{J_1} + \frac{\cos^2 \psi}{J_2} \right), \quad (15)$$

$$g_{23}^{-1} = B \left( \frac{1}{J_1} - \frac{1}{J_2} \right) \cot \Theta \sin \psi \cos \psi, \quad (16)$$

$$g_{33}^{-1} = B \left( \frac{\cos^2 \psi}{J_1} + \frac{\sin^2 \psi}{J_2} \right) \cot^2 \Theta + \frac{B}{J_3}, \quad g_{44}^{-1} = 1, \quad g_{55}^{-1} = \frac{1}{\beta^2}. \quad (17)$$

Using these matrix elements and the value of the determinant from Eq. (10) in the usual Pauli–Podolsky prescription [9]

$$(\nabla\Phi)^i = g^{ij} \frac{\partial\Phi}{\partial x^j}, \quad \nabla^2\Phi = \frac{1}{\sqrt{g}} \partial_i \sqrt{g} g^{ij} \partial_j \Phi, \quad (18)$$

we obtain

$$T = -\frac{\hbar^2}{2B} \nabla^2 = -\frac{\hbar^2}{2B} \left[ \frac{1}{\beta^4} \frac{\partial}{\partial\beta} \beta^4 \frac{\partial}{\partial\beta} + \frac{1}{\beta^2 \sin 3\gamma} \frac{\partial}{\partial\gamma} \sin 3\gamma \frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2} \sum_{k=1,2,3} \frac{Q_k^2}{\sin^2 \left( \gamma - \frac{2}{3}\pi k \right)} \right], \quad (19)$$

where  $Q_k$  are the components of the angular momentum in the intrinsic frame

$$Q_x = -i \left( -\frac{\cos \psi}{\sin \Theta} \frac{\partial}{\partial \Phi} + \sin \psi \frac{\partial}{\partial \Theta} + \cot \Theta \cos \psi \frac{\partial}{\partial \psi} \right), \quad (20)$$

$$Q_y = -i \left( -\frac{\sin \psi}{\sin \Theta} \frac{\partial}{\partial \Phi} + \cos \psi \frac{\partial}{\partial \Theta} - \cot \Theta \sin \psi \frac{\partial}{\partial \psi} \right), \quad Q_z = -i \frac{\partial}{\partial \psi}. \quad (21)$$

## 2.2 Transition to a curved space

The connection between the position-dependent effective mass and curved spaces has been considered in Ref. [8]. According to the findings of Ref. [8], one expects in the present case all elements of the matrix (5) to be divided by  $f^2$

$$g'_{ij} = \frac{g_{ij}}{f^2}, \quad (22)$$

where  $f$  is a function of  $\beta$ , which will appear in the next subsection. As a result, the determinant of the matrix will be

$$g' = \frac{g}{f^{10}}, \quad (23)$$

and the volume element will be

$$dV' = \frac{dV}{f^5}. \quad (24)$$

The elements of the inverse matrix will be

$$g'^{-1}_{ij} = f^2 g^{-1}_{ij}. \quad (25)$$

## 2.3 Equivalence to the deformation-dependent mass case

In Ref. [3] the Bohr equation with a mass depending on the deformation

$$B(\beta) = \frac{B_0}{(f(\beta))^2}, \quad (26)$$

where  $B_0$  is a constant, has been considered, leading to a modified Bohr equation of the form

$$H\Psi = \left[ -\frac{1}{2} \frac{\sqrt{f}}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 f \frac{\partial}{\partial \beta} \sqrt{f} - \frac{f^2}{2\beta^2 \sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} + \frac{f^2}{8\beta^2} \sum_{k=1,2,3} \frac{Q_k^2}{\sin^2 \left( \gamma - \frac{2}{3}\pi k \right)} + v_{eff} \right] \Psi = \epsilon \Psi, \quad (27)$$

where reduced energies  $\epsilon = B_0 E/\hbar^2$  and reduced potentials  $v = B_0 V/\hbar^2$  have been used, with

$$v_{eff} = v(\beta, \gamma) + \frac{1}{4}(1 - \delta - \lambda)f\nabla^2 f + \frac{1}{2}\left(\frac{1}{2} - \delta\right)\left(\frac{1}{2} - \lambda\right)(\nabla f)^2. \quad (28)$$

According to Ref. [8], in order to obtain the Schrödinger equation in the form of Eq. (27), one has to start with the equation

$$H_g \tilde{\Psi} = \left[-\frac{1}{2}\nabla^2 + u_g\right] \tilde{\Psi} = \left[-\frac{1}{2}\frac{1}{\sqrt{g'}}\partial_i \sqrt{g'}g'^{-1}_{ij}\partial_j + u_g\right] \tilde{\Psi}, \quad (29)$$

where

$$\tilde{\Psi} = f^{5/2}\Psi, \quad (30)$$

while reduced energies and reduced potentials are used, as in Eq. (27). The exponent in the last equation is related to the dimensionality of the space.

Substituting the  $g'$  matrix elements and determinant in Eq. (29), and performing the relevant calculation (which closely resembles the pure Bohr case, except for the 44-term), we see that Eqs. (29) and (27) do coincide with

$$u_g = u_{eff} + f\ddot{f} - 2(\dot{f})^2 + 4\frac{f\dot{f}}{\beta}, \quad \dot{f} = \frac{df}{d\beta}, \quad \ddot{f} = \frac{d^2f}{d\beta^2}. \quad (31)$$

## 2.4 Discussion

This result has several important consequences.

1) It becomes clear that solving the Schrödinger equation (27) with deformation dependent mass is equivalent to solving a modified Bohr equation (29) with different metric matrix  $g'$  and another effective potential,  $u_g$ . Between the two equivalent schemes, one chooses to solve Eq. (27) instead of Eq. (29), just because the former can be solved analytically through the use of SUSYQM techniques [10].

2) The wave functions  $\tilde{\Psi} = f^{5/2}\Psi$  are accompanied by the volume element  $dV' = dV/f^5$ . As a result

$$\int \tilde{\Psi}^* \tilde{\Psi} dV' = \int (f^{5/2}\Psi^*)(f^{5/2}\Psi) \frac{dV}{f^5} = \int \Psi^* \Psi dV, \quad (32)$$

i.e., the wave functions  $\Psi$  of the deformation dependent mass problem correspond to the usual Bohr volume element  $dV$ .

3) The simple relation between  $\tilde{\Psi}$  and  $\Psi$  also shows that the wave functions  $\Psi$  satisfy the well-known 24 symmetries of Bohr wave functions [2], which the wave functions  $\tilde{\Psi}$  satisfy by construction. If these symmetries were not satisfied, the solutions could not have been used for the description of nuclei.

The solution of Eq. (27) using SUSYQM techniques [3] is described in this conference by P. E. Georgoudis [11]. The particular case of the Davidson

potential [12]

$$u(\beta) = \beta^2 + \frac{\beta_0^4}{\beta^2}, \quad (33)$$

where the parameter  $\beta_0$  indicates the position of the minimum of the potential, is used. Using SUSYQM techniques it turns out that in this case the Bohr Hamiltonian with mass dependent on the deformation is exactly soluble if the deformation function is of the form

$$f(\beta) = 1 + a\beta^2, \quad a \ll 1. \quad (34)$$

Numerical results for spectra and  $B(E2)$  transition rates for about 50  $\gamma$ -unstable nuclei and 50 deformed nuclei have been obtained [3] and compared to experimental data, with positive results.

### 3 Group contractions

#### 3.1 The $SU(3) \rightarrow [R^5]SO(3)$ contraction

The  $SU(3) \rightarrow [R^5]SO(3)$  contraction [13, 14] is a procedure in which the full  $SU(3)$  algebra, consisting of 8 noncommuting generators, is shrunk into an  $SO(3)$  algebra (consisting of 3 noncommuting generators), accompanied by 5 mutually commuting operators (the quadrupole operators). This simplification occurs in the limit of large boson number in which, in  $SU(3)$ , all intrinsic excitations rise in energy, isolating the ground state band so that  $SU(3)$  goes over, approximately, into a simple rigid rotator. The resulting algebraic structure is, indeed, known [15] to be the algebra of the rigid rotator.

The  $SU(3)$  commutation relations read

$$[\hat{L}_\xi, \hat{L}_\nu] = -\sqrt{2}(1\xi 1\nu | 1\xi + \nu) \hat{L}_{\xi+\nu}, \quad (35)$$

$$[\hat{L}_\xi, \hat{Q}_{SU(3),\nu}^{(2)}] = -\sqrt{6}(1\xi 2\nu | 2\xi + \nu) \hat{Q}_{SU(3),\xi+\nu}^{(2)}, \quad (36)$$

$$[\hat{Q}_{SU(3),\xi}^{(2)}, \hat{Q}_{SU(3),\nu}^{(2)}] = \frac{3}{4} \sqrt{\frac{5}{2}} (2\xi 2\nu | 1\xi + \nu) \hat{L}_{\xi+\nu}. \quad (37)$$

The second order Casimir operator is

$$\hat{C}_2[SU(3)] = \frac{2}{3} \left[ 2\hat{Q}_{SU(3)}^{(2)} \cdot \hat{Q}_{SU(3)}^{(2)} + \frac{3}{4} \hat{L} \cdot \hat{L} \right], \quad (38)$$

while its eigenvalues in the Elliott basis,  $(\lambda, \mu)$ , are

$$C_2(\lambda, \mu) = \frac{2}{3} (\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu). \quad (39)$$

If we consider  $SU(3)$  irreducible representations (irreps) with large values of  $C_2(\lambda, \mu)$ , that is for large boson numbers, we can rescale the quadrupole operator as

$$\hat{q}_{SU(3),\xi}^{(2)} = \frac{\hat{Q}_{SU(3),\xi}^{(2)}}{\sqrt{C_2(\lambda, \mu)}}. \quad (40)$$

The first two commutation relations remain unchanged by the rescaling, while the last one becomes

$$[\hat{q}_{SU(3),\xi}^{(2)}, \hat{q}_{SU(3),\nu}^{(2)}] = \frac{3}{4} \sqrt{\frac{5}{2}} (2\xi 2\nu | 1\xi + \nu) \frac{\hat{L}_{\xi+\nu}}{C_2(\lambda, \mu)}. \quad (41)$$

Then in the limit of large values of  $C_2(\lambda, \mu)$  one gets

$$[\hat{q}_{SU(3),\xi}^{(2)}, \hat{q}_{SU(3),\nu}^{(2)}] = 0. \quad (42)$$

This result, which is obtained for large boson number, is called the contraction of  $SU(3)$  to  $[R^5]SO(3)$ , where  $[R^5]SO(3)$  is the algebra of the rigid rotator [15], generated by the angular momentum operators of  $SO(3)$  and the five commuting operators  $\hat{q}_{SU(3),\xi}^{(2)}$ ,  $\xi = -2, -1, 0, 1, 2$ .

An immediate consequence of Eqs. (41) and (42) is that, in the contraction limit, terms proportional to the angular momentum  $\hat{L}$  can be ignored. In the IBA framework, in which  $\hat{L}$  is proportional to  $(d^\dagger \tilde{d})^{(1)}$ , this implies that  $(d^\dagger \tilde{d})^{(1)}$  terms can be ignored.

In the limit of large values of  $C_2(\lambda, \mu)$  and  $\lambda \geq \mu$  the intrinsic quadrupole moments become [14]

$$q_0 = \frac{1}{2\sqrt{2}}(2\lambda + \mu + 3), \quad q_2 = \frac{1}{4} \sqrt{3(\mu - K)(\mu + K + 2)}, \quad (43)$$

where  $K$  is the eigenvalue of the angular momentum projection on the body-fixed  $z$ -axis, for which  $K \leq L$  is valid, as one can see from the algorithm of the  $SU(3) \supset SO(3)$  reduction [1]. For states with  $\lambda \gg L$  (therefore also  $\lambda \gg K$ ) and  $\lambda \gg \mu$  one then obtains [13]

$$q_0 = \frac{\lambda}{\sqrt{2}}, \quad (44)$$

while  $q_2$  becomes negligible. Since the ground state band belongs to the  $(2N, 0)$  irreducible representation (irrep) of  $SU(3)$ , while other low-lying bands belong to irreps  $(2N - 4i - 6j, 2i)$ ,  $i = 0, 1, 2, \dots$ ,  $j = 0, 1, 2, \dots$  with relatively low  $i$ ,  $j$ , the contraction occurs in the large  $N$  limit. Thus in the case of interest the intrinsic quadrupole moment becomes

$$q_0 = N\sqrt{2}. \quad (45)$$

An equivalent statement is that one can approximately replace the operator  $\hat{Q}_{SU(3)}^{(2)}$  by the scalar  $\lambda/\sqrt{2}$ , as one can see from Eqs. (38) and (39), since the terms containing  $\hat{L}$  and  $\mu$  are negligible in this limit, having as a consequence that only the first term in the rhs of these equations survives. A formal justification for this replacement is given in Ref. [5], where matrix elements of the commutators of the relevant parts of the Hamiltonian with the quadrupole operator are properly considered, resulting in the appearance of the intrinsic quadrupole moment.

It should be noticed that the above results have been obtained in irreps with  $\lambda \gg L$ , thus they regard the low lying part of the spectrum.

### 3.2 $\overline{\text{SU}(3)}$

In  $\text{SU}(3)$  the irreps are built out of the (2,0) irrep, while in the case of  $\overline{\text{SU}(3)}$  [1] the irreps are built out of the (0,2) irrep [1]. As a result, in the  $\overline{\text{SU}(3)}$  framework one is interested in states with large values of  $C_2(\lambda, \mu)$  and  $\lambda < \mu$ , in which the intrinsic quadrupole moments become [14, 16]

$$q_0 = -\frac{1}{2\sqrt{2}}(\lambda + 2\mu + 3), \quad q_2 = -\frac{1}{4}\sqrt{3(\lambda - K)(\lambda + K + 2)}. \quad (46)$$

For states with  $\mu \gg L$  and  $\mu \gg \lambda$  one then obtains

$$q_0 = -\frac{\mu}{\sqrt{2}}, \quad (47)$$

while  $q_2$  becomes negligible. Since the ground state band belongs to the (0, 2N) irrep of  $\overline{\text{SU}(3)}$ , while other low-lying bands belong to irreps  $(2i, 2N - 4i - 6j)$ ,  $i = 0, 1, 2, \dots, j = 0, 1, 2, \dots$  with relatively low  $i, j$ , the contraction does occur in the large  $N$  limit, the intrinsic quadrupole moment becoming

$$q_0 = -N\sqrt{2}. \quad (48)$$

Since  $\text{SU}(3)$  is associated to prolate shapes, while  $\overline{\text{SU}(3)}$  is related to oblate shapes, the signs in Eqs. (45) and (48) are consistent with the fact that intrinsic quadrupole moments are known to be positive for prolate nuclei and negative for oblate nuclei [7].

The first contraction has been used in Ref. [5] for determining a line of approximate  $\text{SU}(3)$  symmetry inside the symmetry triangle of the IBA, as described by S. Karampagia in this conference [17].

### 3.3 The $\text{O}(6) \rightarrow [\text{R}^5]\text{SO}(5)$ contraction

A procedure similar to that of subsection 3.1 is followed in the contraction of  $\text{O}(6)$  to  $[\text{R}^5]\text{SO}(5)$  [18, 19]. This is a procedure in which the full  $\text{O}(6)$  algebra, consisting of 15 noncommuting generators, is shrunk into an  $\text{SO}(5)$  algebra (consisting of 10 noncommuting generators), accompanied by 5 mutually commuting operators (the quadrupole operators). The resulting algebraic structure is known [19] to be the algebra of the  $\gamma$ -unstable rotor.

The commutation relation for the quadrupole operators reads

$$[\hat{Q}_{\text{O}(6),\xi}^{(2)}, \hat{Q}_{\text{O}(6),\nu}^{(2)}] = 2 \sum_{k=1,3} (2\xi 2\nu | k\xi + \nu) (d^\dagger \tilde{d})_{\xi+\nu}^{(k)}. \quad (49)$$

The second order Casimir operator is [1]

$$\hat{C}_2[\text{O}(6)] = 2\hat{Q}_{\text{O}(6)}^{(2)} \cdot \hat{Q}_{\text{O}(6)}^{(2)} + 4 \sum_{k=1,3} (d^\dagger \tilde{d})^{(k)} \cdot (d^\dagger \tilde{d})^{(k)}. \quad (50)$$

Its eigenvalues are

$$C_2(\sigma) = 2\sigma(\sigma + 4), \quad (51)$$

where  $\sigma$  is the quantum number characterizing the irreps of  $O(6)$ .

If we consider  $O(6)$  irreps with large  $\sigma$ , we can rescale the quadrupole operator as

$$\hat{q}_{O(6),\xi}^{(2)} = \frac{\hat{Q}_{O(6),\xi}^{(2)}}{\sqrt{C_2(\sigma)}}. \quad (52)$$

Then the commutation relation for the quadrupole operators becomes

$$[\hat{q}_{O(6),\xi}^{(2)}, \hat{q}_{O(6),\nu}^{(2)}] = 2 \sum_{k=1,3} (2\xi 2\nu | k\xi + \nu) \frac{(d^\dagger \tilde{d})_{\xi+\nu}^{(k)}}{C_2(\sigma)}. \quad (53)$$

Then in the limit of large  $\sigma$  (and small  $\tau$ , where  $\tau$  is the quantum number characterizing the irreps of  $O(5)$ ) [19] one gets

$$[\hat{q}_{O(6),\xi}^{(2)}, \hat{q}_{O(6),\nu}^{(2)}] = 0. \quad (54)$$

This procedure is called the contraction of  $O(6)$  to  $[R^5]SO(5)$ , where  $[R^5]SO(5)$  is the algebra of the  $\gamma$ -unstable rotator, generated by the operators of  $SO(5)$  and the five commuting operators  $q_{O(6),\xi}^{(2)}$ ,  $\xi = -2, -1, 0, 1, 2$ , which are the coordinates [19].

An immediate consequence of Eqs. (53) and (54) is that, in the contraction limit, terms proportional to  $(d^\dagger \tilde{d})^{(k)}$  can be ignored.

The most leading  $O(6)$  irrep, to which the ground state band belongs, is  $(N)$ . Thus in the large boson number limit it is appropriate to use this contraction. The intrinsic quadrupole moment will then be

$$q'_0 = \sigma, \quad (55)$$

as can be seen from Eqs. (50) and (51). Thus in the case of interest the intrinsic quadrupole moment becomes

$$q'_0 = N. \quad (56)$$

It should be noticed that the above results have been obtained in irreps with  $\sigma \gg \tau$ , thus they regard the low lying part of the spectrum (since  $L \leq 2\tau$ , as seen from the algorithm of the  $SO(5) \supset SO(3)$  reduction [1]).

### 3.4 Application

The first contraction has been used in Ref. [5] for determining a line of approximate  $SU(3)$  symmetry inside the symmetry triangle of the IBA model, as described by S. Karampagia in this conference [17]. This line lies very close to the Alhassid–Whelan arc of regularity [20, 21], corroborating an underlying approximate  $SU(3)$  symmetry [22] as the reason for the existence of this region of regularity among chaotic regions in the symmetry triangle of the IBA model.

## 4 Conclusion

We have shown how conformal maps can be used for introducing a deformation-dependent mass in the Bohr collective model, as well as how group contractions can be used for determining approximate symmetries within the symmetry triangle of the Interacting Boson Approximation model.

## References

- [1] F. Iachello, A. Arima, *The Interacting Boson Model*, Cambridge University Press, Cambridge (1987).
- [2] A. Bohr, *Mat. Fys. Medd. K. Dan. Vidensk. Selsk.* **26** no. 14 (1952).
- [3] D. Bonatsos, P.E. Georgoudis, D. Lenis, N. Minkov, C. Quesne, *Phys. Rev. C* **83** (2011) 044321.
- [4] P. Ring, P. Schuck, *The Nuclear Many-Body Problem*, Springer, Berlin (1980).
- [5] D. Bonatsos, S. Karampagia, R.F. Casten, *Phys. Rev. C* **83** (2011) 054313.
- [6] E. Inónú, E.P. Wigner, *Proc. Natl. Acad. Sci. (N.Y.)* **39** (1953) 510.
- [7] R.F. Casten, *Nuclear Structure from a Simple Perspective*, Oxford University Press, Oxford (1990).
- [8] C. Quesne, V.M. Tkachuk, *J. Phys. A: Math. Gen.* **37** (2004) 4267.
- [9] B. Podolsky, *Phys. Rev.* **32** (1928) 812.
- [10] F. Cooper, A. Khare, U. Sukhatme, *Supersymmetry in Quantum Mechanics*, World Scientific, Singapore (2001).
- [11] D. Bonatsos, P.E. Georgoudis, D. Lenis, N. Minkov, C. Quesne, these proceedings.
- [12] P.M. Davidson, *Proc. R. Soc. London Ser. A* **135** (1932) 459.
- [13] R. Le Blanc, J. Carvalho, D.J. Rowe, *Phys. Lett. B* **140** (1984) 155.
- [14] D.J. Rowe, *Prog. Part. Nucl. Phys.* **37** (1996) 265.
- [15] H. Ui, *Prog. Theor. Phys.* **44** (1970) 153.
- [16] D.J. Rowe, M.G. Vassanji, J. Carvalho, *Nucl. Phys. A* **504** (1989) 76.
- [17] D. Bonatsos, S. Karampagia, R.F. Casten, these proceedings.
- [18] J. Meyer-ter-Vehn, *Phys. Lett. B* **84** (1979) 10.
- [19] J.P. Elliott, P. Park, J.A. Evans, *Phys. Lett. B* **171** (1986) 145.
- [20] Y. Alhassid, N. Whelan, *Phys. Rev. Lett.* **67** (1991) 816.
- [21] N. Whelan, Y. Alhassid, *Nucl. Phys. A* **556** (1993) 42.
- [22] D. Bonatsos, E.A. McCutchan, R.F. Casten, *Phys. Rev. Lett.* **104** (2010) 022502.