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# Symplectic Extensions of the Proton-Neutron Version of the Interacting Boson Model

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**Abstract.** We introduce the symplectic extension $Sp(24, R) \supset U(12) \supset U_\pi(6) \otimes U_\nu(6)$ of the proton ($\pi$)-neutron ($\nu$) version of the prominent Interacting Boson Model (IBM - 2). We consider in particular a new reduction chain starting with the direct product $Sp(4, R) \otimes O(6)$, where the $Sp(4, R)$ group is used as a classification group for the even - even nuclei specified by the total number of bosons $N$, that build them and the third projection $F_0$ of the $F-$spin. This allows for a unified description of sequences of nuclei with a general Hamiltonian in which the interaction strengths, depend on the classification quantum numbers and interactions are expressed in terms of the Casimir invariants of all the subgroups of $O(6)$, defining the dynamical symmetry.

## 1 Introduction

Since the introduction of the first version of the Interacting Boson Model /IBM -1/ [1] the description of the nuclear collective degrees of freedom in terms of bosons has proved as very appropriate and is widely used by experimentalist and theoreticians as an effective tool for the description of the rather complex and rich nuclear spectra and transitions in particular in heavy nuclei. The mathematical tool that is used is based on the algebraic constructions and their irreducible representations (irreps) and proved as very efficient in its applications. It introduces the notion of dynamical symmetries, represented by chains of group-subgroup structures. The latter provide the corresponding phenomenological models simultaneously with a complete orthonormal basis, labelled by the quantum numbers of the algebraic irreps of the groups from the chain and respective Hamiltonians, expressed in terms of their Casimir invariants. This leads to exactly solvable limiting cases, which provide the benchmarks for the different types of collective behavior of the many-body system.

In the IBM - 1 [1] model an **s** ($l = 0$) and a **d** ($l = 2, m = \pm 2, \pm 1, 0$) bosons are introduced as building blocks of the $U(6)$ algebra, reduced to the $SO(3)$ -

algebra of the angular momentum by means of the reduction chains:

$$\begin{array}{ccccccccc} & \nearrow & U(5) & \longrightarrow & SO(5) & \searrow & & \\ U(6) & \longrightarrow & O(6) & \longrightarrow & SO(5) & \longrightarrow & SO(3) \\ & \searrow & U(3) & \longrightarrow & SU(3) & \nearrow & & \end{array} \quad (1)$$

It has been shown in various applications [2] that each of the reduction chains in the scheme (1) corresponds to a distinctive geometrical interpretation of the nuclear shape. The one trough $U(5)$ is related to the spectra of vibrational (nearly spherical) nuclei, the $O(6)$ limit to the $\gamma-$unstable ones and $U(3)$ to rotational (axially deformed) nuclei.

In this work, we aim at the description not of a single nucleus, but of sets of nuclei, in a way that reflects the development of collectivity in them with the change of the number of bosons, representing pairs of valence particles. This is achieved by introducing the symplectic extension of the boson realization of compact unitary algebras employed in the dynamical symmetries of the IBM. This approach will be illustrated on the example of IBM-2 version of the model, where the proton and neutron (charge) degrees of freedom, forming two separate spaces are introduced in [3]. In this way, the link to the underlying single-particle shell structure is more direct and transparent.

## 2 Reduction Scheme for the Boson Representations of the *Sp*(4*k,R*) Algebras

In general the boson representation of the $Sp(4k, R)$ algebra is realized in terms of bilinear combinations of creation $a^\dagger_{\alpha i}$ and annihilation $a_{\alpha i}$ operators with two indexes, $\alpha = 1, 2;\ i = 1, 2, ..., k$, that satisfy Bose commutation relations:

$$[a_{\alpha i}, a^\dagger_{\beta j}] = \delta_{\alpha\beta}\delta_{ij} \quad (2)$$

(all other commutators are zero). Specifically, the symplectic extension of the $u(2k)$ algebra to the $Sp(4k, R)$ algebra is obtained [4], by appending the operators

$$a^\dagger_{\alpha i}a^\dagger_{\beta j},\ \ a_{\alpha i}a_{\beta j},\ \ \alpha, \beta = 1, 2;\ i, j = 1, ..., k \quad (3)$$

to its Weyl generators $a^\dagger_{\alpha i}a_{\beta j}$.

In the considered case, as the Greek indices take only two possible values, it is easy to see that they actually label the two dimensional spinor representation of $su(2)$, with projections $\alpha = \pm 1/2$. In most of the applications, the tensor properties of the operators $a^\dagger_{\alpha i}$ and $a_{\alpha i}$ and their bilinear products (3) in respect to the $so_L(3)$ algebra are considered, because the components of the angular momentum operators that generate it are the main observable in the theory of nuclear structure. The relation of the "space" indexes $i, j$, which take $k$-values to the indexes $l, m$ is not straight forward, but it is important to say that the number of possible values of the projections $m$ of the corresponding $l$ should give the value of $k$. For example in the IBM - 1 [1] model the $s$–boson with $l = 0, m = 0$ and $d$–boson with $l = 2, m = 0, \pm 1, \pm 2$ represent the $k = 6$ case and the construction considered in this work can be applied in its proton-neutron version (IBM-2) $\alpha = \pm 1/2$ for protons and neutrons respectively [3] or for particles or holes in its particle-hole version [5]. In terms of double tensors $u^{\dagger\, lm}_{t\tau}$ and $u^{lm}_{t\tau}$ in respect to the angular momentum $l$ and a t-spin $t$, the generators of the $sp(4k, R)$ algebra with $k = (2l + 1))$ in the general case are [6]:

$$F^{LM}_{TT_0} = (G^{LM}_{TT_0})^{\dagger} = \frac{1}{\sqrt{2}} \sum_{m_1\, m_2\, \tau_1\, \tau_2} C^{LM}_{lm_1,\, lm_2} C^{TT_0}_{t\tau_1,\, t\tau_2} u^{\dagger\, lm_1}_{t\tau_1} u^{\dagger\, lm_2}_{t\tau_2}$$

$$= \frac{1}{\sqrt{2}} \left( u^{\dagger\, l}_{t} \otimes u^{\dagger\, l}_{t} \right)^{LM}_{TT_0} \tag{4a}$$

$$A^{LM}_{TT_0} = \sum_{m_1\, m_2\, \tau_1\, \tau_2} C^{LM}_{lm_1,\, lm_2} C^{TT_0}_{t\tau_1,\, t\tau_2} u^{\dagger\, lm_1}_{t\tau_1} u^{lm_2}_{t\tau_2} (-1)^{l-m_2} (-1)^{t-\tau_2}$$

$$= \left( u^{\dagger\, l}_{t} \otimes \tilde{u}^{l}_{t} \right)^{LM}_{TT_0} \tag{4b}$$

where

$$\tilde{u}^{lm}_{t\tau} = (-1)^{l+m} (-1)^{t+\tau} u^{l\, -m}_{t\, -\tau} \ . \tag{5}$$

The symmetry properties of the Clebsch–Gordan coefficients $C^{LM}_{lm,\, ln}$ and $C^{T\tau}_{t\tau_1,\, t\tau_2}$ result in some restrictions on the possible values of the angular momentum $L$ and isospin $T$ of two-boson generators $F^{LM}_{Tt}$ and $G^{LM}_{Tt}$ (4a), namely

$$(-1)^{2l-L} (-1)^{2t-T} = 1 \ . \tag{6}$$

The commutation relations between the double tensors (4a) and (4b) are given in [6] and from them it follows that they define an algebra of the non-compact symplectic group $Sp(4k, R)$.

The boson representation of the $Sp(4k, R)$ algebra is a reducible one that decomposes into two irreducible representations. One of them acts in the space $\mathcal{H}_+$, spanned over the vectors for which the number of bosons $n$ is even, and the other acts in $\mathcal{H}_-$ defined by the condition $n$ – odd so that $\mathcal{H} = \mathcal{H}_+ + \mathcal{H}_-$. By construction, each of the subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ spans a reducible representation of $U(2k)$ which decomposes into a direct sum of eigensubspaces of the first Casimir invariant $\mathcal{N} = \sum a^{\dagger}_{\alpha i} a_{\alpha i}$ of $U(2k)$. In this way the totally symmetric irreducible unitary representation (IUR) of $U(2k)$, denoted by $[n]_{2k}$ is realized.

Therefore, the group $U(2k)$ appears as a maximal compact subgroup, which further contains the direct product $U(2) \otimes U(k)$ of the two mutually complementary subgroups generated by the operators $F_{\alpha\beta} = \sum a^{\dagger}_{\alpha i} a_{\beta i}$ and $A_{ij} = \sum a^{\dagger}_{\alpha i} a_{\alpha j}$, respectively. The operator $\mathcal{N} = F_{\alpha\alpha} = A_{ii}$ is the first-order Casimir operator for the groups $U(2)$ as well as $U(k)$. Because the groups $U(2)$ and $U(k)$ are mutually complementary and the representations $[n]_{2k}$ are symmetric, all the Casimir operators of $SU(k)$ can be expressed in terms of $\mathcal{N}$ and the second-order Casimir operator $\mathbf{F^2}$ of $SU(2)$,

$$C_2^{(k)} = 2\mathbf{F^2} + (\mathbf{k-2})\mathcal{N} + \frac{(\mathbf{k-2})}{\mathbf{2k}}\mathcal{N}^{\mathbf{2}}. \tag{7}$$

Next, the IURs of the groups $SU(2)$, $SU(k)$, and $SU(2) \otimes SU(k)$ at $n$ = fixed, can be labelled by the eigenvalues $F(F+1)$ ($F \equiv T$) of the operator $\mathbf{F^2}$, where $F = n/2, n/2-1, ..., 0$ *or* $1/2$ for $n$ even or odd, respectively. Thus when $n$ is fixed and $F$ is fixed, $2F+1$ equivalent representations of the group $SU(k)$ arise. Each of them is labelled by the eigenvalues of the operator $F_0$: $-F, -F+1, \ldots, F$. The above reduction rules follow from the decomposition of the totaly symmetric $U(2k)$ irreps into the equivalent irreps of the direct product $SU(2) \otimes U(k)$ labelled by two-rowed Young tableaux:

$$[n]_{2k} = \sum_{i=0}^{<n/2>} [n-i, i]_k \, . \, [n-2i]_2 \tag{8}$$

Hence, in the framework of the discussed boson representation of the $Sp(4k, R)$ algebra all possible irreducible representations of the group $SU(k)$ are determined uniquely through all possible sets of the eigenvalues of the Hermitian operators $\mathbf{N}$, $\mathbf{F^2}$ and $\mathbf{F_0}$. On the other hand, the so called ladder representation of the noncompact group $U(k,k)$ acts in the space of the boson representation of the $Sp(4k, R)$ algebra. There exists a connection between this ladder representation and the boson representation of $U(2k)$, which is realized through the third generator $F_0$ of the multiplier $SU(2)$ of the already mentioned direct product. This operator is also the first Casimir operator of the group $U(k,k)$. Different aspects of this relationship will be revealed in more details in the applications. As shown in [4], both reduction chains

$$Sp(4k, R) \supset U(2k) \supset U(2) \otimes U(k) \supset SU(k) \tag{9}$$

$$Sp(4k, R) \supset U(k,k) \supset U(k) \otimes U(k) \supset SU(k) \tag{10}$$

are equally convenient for the description of the representations of the final group $SU(k)$.

## 3 The *Sp*(4,*R*) (*k* = 1 Case) Classification Scheme

The simplest $k = 1$ case, illustrates in a very clear way the algebraic construction that can be used for the classification of the even-even nuclear systems. Similar methods are used for the classification of elementary particles. The generators of $Sp(4, R)$ algebra are realized in terms of two types of one-dimensional (scalar) creation $(\pi^\dagger, \nu^\dagger)$ and annihilation $(\pi, \nu)$ operators. The reduction of the boson representation of the classification group $Sp(4, R)$ to its compact $u(2)$ (9) and non-compact $u(1,1)$ (9) subalgebras [7],

$$\begin{array}{ccccc} & N_t \nearrow & u(2) & \searrow F_0 & \\ sp(4,R) & & & & u_\pi(1) \oplus u_\nu(1), \\ & F_0 \searrow & u(1,1) & \nearrow N_t & \end{array} \tag{11}$$

is the mathematical underpinning of the scheme. As illustrated by (11), the reduction is realized by means of the operator that counts the total number of particles, $N_t = (N_\pi + N_\nu)$ ($N_\pi = \pi^\dagger\pi, N_\nu = \nu^\dagger\nu$ ) which is the first order invariant of $u(2)$, and the operator of the third projection of the $F$-spin, $F_0 = \frac{1}{2}(N_\pi - N_\nu)$, which does not differ essentially from the first order Casimir of $u(1,1)$. $N_t$ reduces the space $\mathcal{H}$, in which the boson representation of $sp(4,R)$ acts, into a direct sum of a totally symmetric irreducible unitary representations /IUR/ of $su(2)$, labelled by $N_t = 0, 2, 4, ...$ (even $\mathcal{H}_+$) or $N_t = 1, 2, 3, ...$ (odd $\mathcal{H}_-$). The operator $F_0$ reduces the space $\mathcal{H}$ to the ladder series of $u(1,1)$, defined by its fixed eigenvalues. The same operator $F_0$ reduces each $u(2)$ representation (fixed value of $N_t$) to the representations of $u_\pi(1) \oplus u_\nu(1)$ labelled by $N_\pi$ and $N_\nu$, respectively. The same is obtained by reducing the $u(1,1)$ ladders with the operator $N_t$.

The relation of the algebraic operators used in the classification scheme to the nuclear characteristics in the valence shell, is quite natural when $N_\pi = \frac{1}{2}(N_p - Z^{(1)})$ and $N_\nu = \frac{1}{2}(N_n - N^{(1)})$ are counted as the numbers of proton and neutron valence pairs of the nucleus from a given shell, in which $Z^{(1)}$

Table 1. Nuclei from the $(50, 82|82, 126)_-$ shell mapped on the $\mathcal{H}_-$ ($N_t$-odd) subspace of $Sp(4,R)$. Each nucleus is defined by the total number of valence bosons $N$ or boson holes $\bar{N}_t$, which label the rows from the left and from the right side respectively and the third projections $F_0$ ($\bar{F}_0$) of the $F-$ spin, which label the columns on the top and bottom, respectively.

| | | $F_0$ | | | |
|---|---|---|---|---|---|
| $N$ | $5/2$ | $3/2$ | $1/2$ | $-1/2$ | $\bar{N}$ |
| 1 | | | $^{134}$Te | $^{134}$Sn | 37 |
| 3 | | $^{138}$Ba | $^{138}$Xe | $^{138}$Te | 35 |
| 5 | $^{142}$Nd | $^{142}$Ce | $^{142}$Ba | $^{142}$Xe | 33 |
| 7 | $^{146}$Sm | $^{146}$Nd | $^{146}$Ce | $^{146}$Ba | 31 |
| 9 | $^{150}$Gd | $^{150}$Sm | $^{150}$Nd | $^{150}$Ce | 29 |
| 11 | $^{154}$Dy | $^{154}$Gd | $^{154}$Sm | $^{154}$Nd | 27 |
| 13 | $^{158}$Er | $^{158}$Dy | $^{158}$Gd | $^{158}$Sm | 25 |
| 15 | $^{162}$Yb | $^{162}$Er | $^{162}$Dy | $^{162}$Gd | 23 |
| 17 | $^{166}$Hf | $^{166}$Yb | $^{166}$Er | $^{166}$Dy | 21 |
| 19 | $^{170}$W | $^{170}$Hf | $^{170}$Yb | $^{170}$Er | 19 |
| 21 | $^{174}$Os | $^{174}$W | $^{174}$Hf | $^{174}$Yb | 17 |
| 23 | $^{178}$Pt | $^{178}$Os | $^{178}$W | $^{178}$Hf | 15 |
| 25 | $^{182}$Hg | $^{182}$Pt | $^{182}$Os | $^{182}$W | 13 |
| 27 | $^{186}$Pb | $^{186}$Hg | $^{186}$Pt | $^{186}$Os | 11 |
| 29 | | $^{190}$Pb | $^{190}$Hg | $^{190}$Pt | 9 |
| 31 | | | $^{194}$Pb | $^{194}$Hg | 7 |
| 33 | | | | $^{198}$Pb | 5 |
| 35 | | | | | 3 |
| 37 | | | | | 1 |
| $N$ | $-11/2$ | $-9/2$ | $-7/2$ | $-5/2$ | $\bar{N}$ |
| | | $\bar{F}_0$ | | | |

and $N^{(1)}$ are the numbers of protons and neutrons of the double magic nucleus at the beginning of the shell. Then $N_t$ and $F_0$ are exactly the operators reducing the $sp(4, R)$ spaces, and their interpretation corresponds to the one of the Interacting Boson Model - 2 (IBM-2) [8], as the total number of valence bosons and the third projection of the $F$-spin. This is illustrated on Table 1 for the even-even nuclei from the major shell with $50 < Z < 82; 82 < N < 156$, labelled as $(50, 82|82, 126)_{-}$. This classification of the nuclei from the major nuclear shells [9] was proven very useful in the empirical investigation of the behavior of important collective nuclear characteristics, like the energies of the low-lying states of the even-even nuclei [7] or the semiempirical "collective masses" [10]. In the cases of a smooth behavior of these variables, analytic formulae for their behavior as functions of the classification quantum numbers were obtained describing the investigated data for large amount of classified nuclei.

Another good example of the construction and use of the symplectic extension of the boson models of nuclear structure is the $k = 3$ case, where the dynamical symmetry $U(6) \supset Sp(12, R)$ of the Interacting Vector Boson Model (IVBM), that is constructed by means of two types of vector bosons [11]. The model is applied for the description of the collective modes and their interactions in heavy even-even nuclear systems. This extension yields a rich subgroup structure of $Sp(12, R)$ within which some new non-compact subgroup structures appear. The reduction through the direct product $Sp(4, R) \otimes SO(3)$ [12] is of particular importance as it is not only used to describe sequences of states with fixed angular momentum, but also to elucidate the connection of this chain with the other dynamical symmetries of the model through its $SU(2)$ and $SU(1, 1)$ subgroups that are also substantial parts of the two other dynamical symmetries considered. In this way it plays the role of a generalized group classification scheme, one that orders (distributes) the collective excitations in spectra of individual nuclei in terms of the collective (boson) structure of their band-head configurations.

## 4 The Symplectic Extension of the Interacting Boson Model – 2

The version of the model, denoted as IBM-2, that considers the proton and neutron nuclear subsystems has dynamical symmetry represented by the direct product of the two algebras $U_\pi(6) \otimes U_\nu(6)$, which is obviously contained in $U(12)$ [13]. Based on the the general reduction of the $Sp(4k, R)$ [4] algebra given in (9) and (10) it is straightforward to realize the symplectic extension of the group of dynamical symmetry of the model $U(12) \supset Sp(24, R)$ [9], as the $k = 6$ case. Further making use of the simple classification properties of the $Sp(4, R)$ group, outlined in Section 3, and the interpretation of the reduction operators in terms of the IBM-2 [8], we explore an other possible reduction [14] of $Sp(4k, R)$ algebra – through its noncompact subalgebra $sp(4, R)$

$$Sp(4k, R) \supset Sp(4, R) \otimes SO(k). \tag{12}$$

In the considered case, we make use of the following correspondence between the two chains (trough $u(12)$ (9) and $sp(4,R)$ (12)) of subalgebras of $sp(24,R)$:

$$\begin{array}{lllll} sp(24,R) & \supset & sp(4,R) & \otimes & so(6) \\ \cup & & \cup & & \cap \\ u(12) & \supset & u(2) & \otimes & su(6), \end{array} \tag{13}$$

which plays an important role [15]. Result (13) is a consequence of the equivalence of the $u(2) \supset sp(4,R)$ algebra of $F-$ spin, distinguishing the proton and neutron valence particles in IBM-2 [8] and which is complementary to the $su(6)$ in the reduction $u(2) \otimes su(6) \subset u(12)$.

Using the general reduction scheme given by the chains (9) and (10) in conjunction with the reduction (12) and the correspondence (13) we obtain the symplectic extension of the IBM-2:

$$\begin{array}{ccccc} & \nearrow & U_p(6) \otimes U_n(6) & \searrow & \\ U(12) & \longrightarrow U(2) & \otimes & U_{p+n}(6) & \\ \uparrow & & \uparrow & \downarrow & \\ Sp(24,R) & \longrightarrow Sp(4,R) \otimes & & O(6) & \\ \downarrow & & \downarrow & \uparrow & \\ U(6,6) & \longrightarrow U(1,1) \otimes & & U^*_{p-n}(6) & \\ & \searrow & U_p(6) \otimes U^*_n(6) & \nearrow & \end{array} \tag{14}$$

The formal algebraic aspects of this construction are in a close analogy with the symplectic extension of the $k=3$ case of the Interacting Vector Boson Model and details on it can be found in [11]. From the $u(6)$ algebra down to the algebra of the angular momentum one can proceed with the reductions defining the three limiting cases of IBM-1 [1] through the $U(5)$, $U(3)$ and $O(6)$ (1), which have the anharmonic vibrator, the axial rotor and the $\gamma$-unstable rotor as geometrical analogs. Actually the $O(6)$ multiplier of the algebra of $Sp(4,R)$ on the second row of (14) is exactly the one that will start the respective limit of the model. It was shown in [16] that using an infinite dimensional algebraic technique based on the relationship of the quantum numbers of representations and the second order Casimir invariants of the bases $U(5) \supset SO(5)$ and $SO(6) \supset SO(5)$ with the $SU^d(1,1) \supset U(1)$ and the $SU^{sd}(1,1) \supset U(1)$ respectively, exact analytic solutions can be obtained for the $O(6) \leftrightarrow U(5)$ transitional cases.

In this reduction scheme (14) we have, as for the $k=3$ case of the IVBM, the vertical structure in the reduction of $Sp(4,R)$ (11), but in this case related to the classification of all the even-even nuclei from a given major nuclear shell. This follows from the physical interpretation of the reduction operators as the operators of the total number of valence bosons (proton and neutron pairs) – $N = (N_\pi + N_\nu)$, the valence isospin – $F = \frac{N}{2}, \ldots, |\frac{N_\pi - N_\nu}{2}|$ and its third projection $F_0 = \frac{1}{2}(N_\pi - N_\nu)$. This construction involving a classification group in a larger dynamical group allows us to treat in a unified way the properties of sequences of nuclei. Furthermore, a way that is similar to the considerations in [16], it allows us to obtain analytic solutions for the energy spectrum and the transition operators of sequences of nuclei defined by the classification quantum numbers.

In this regard we are motivated by the empirical investigation of the experimental energies of the ground state bands of all even-even nuclei with $A > 20$, [7], based on the $Sp(4, R)$-classification scheme [9]. This reveals a smooth and periodic behavior of the energies as classified in fixed $F_0$ multiplets for changing values of $N_t$. In a more consistent application of the reduction scheme (14), the interactions can be prescribed by the respective dynamical symmetry, as suggested in [7], which yields a generalized phenomenological description by means of the Hamiltonian:

$$H = a_6 C_2[O(6)] + a_5 C_2[O(5)] + b_3(N) C_2[O(3)], \qquad (15)$$

where the inertial parameter $b_3$ is evaluated as a function of the classification quantum number $b_3(N_t) = b_3/(aN_t + bN_t^2)$, when we consider nuclei in a $F-$ spin multiplet ($F_0-$fixed). We illustrate this approach in Figure 1 for ground state bands'(gsb) energies of the sequence of nuclei from the $F_0 = \frac{5}{2}$ column of the shell $(50, 82|82, 126)$ given on Table 1. The mixed symmetric (two-rowed)

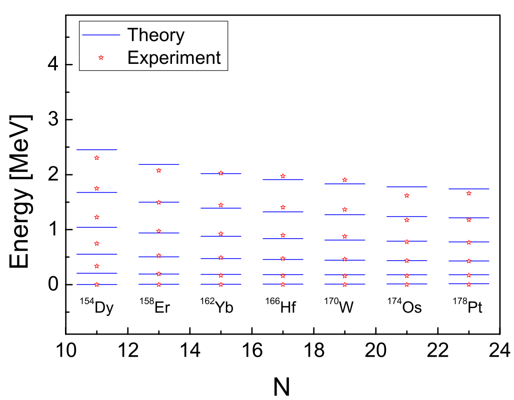


Figure 1. Comparison between the experimental(lines) and theoretical (stars) values of the gsb energies of the sequence of nuclei from the $F_0 = \frac{5}{2}$ column of the shell $(50, 82|82, 126)_-$.

states (if considered) are pushed up by the first term. In the fitting procedure the experimental energies up to L=10 of the considered nuclei are compared with the theoretical predictions for the basis states that belong to the fully symmetric irrep of O(6). This is a standard choice, since when $a_6 > 0$, this irrep gives the lowest energy values of the states. The non-symmetric (mixed symmetry) irreps will be pushed up in energy and can be used eventually for the description of other low-lying excited bands. An alternative parametrization, where $b_3(N_t) = (1+aN_t+bN_t^2)$ gives equivalent results (the same $\chi^2$) , and in the $N \to \infty$ limit this term does not disappear. The obtained results are quite good in particular for the low-lying states, keeping in mind the quite big amount of reproduced experimental data.

## 5 Conclusions

Based on general reduction schemes for boson representations of symplectic algebras of the type $Sp(4k, R)$ [4], we first present applications of the simplest $k = 1$ case. Specifically, we showed that the $Sp(4, R)$ algebra is very convenient for classifying many-body nuclear systems within a major nuclear shell when we use an interpretation of the reduction operators in terms of IBM-2 bosons ( [13]), representing pairs of valence protons and neutrons, the constituent particles of any nuclei. This interpretation was advanced for a generalized description [7], [9] of nuclear properties based on the quantum labels of the associated classification scheme.

We presented a generalized reduction scheme for the symplectic extension $(k = 6)$ of the proton-neutron version of the IBM-2 [13]. A point of interest in this case is that the $k = 1$ results reappear with the same interpretation of the $Sp(4, R)$ structure and reductions as discussed in [9]. This limit of the theory provides us with an opportunity to describe, within the framework of the $Sp(24, R)$ dynamical symmetry, the development of collectivity across an entire nuclear shell, allowing for an investigation of transitions between the different limiting cases, while at the same time retaining the strategic advantage of dynamical symmetries for obtaining exact analytic solutions. This approach can also provide for an algebraic evaluation of critical point features, such as phase/shape transitions, in terms of the nuclear characteristics employed in their classification.

In summary, we explored the richer possibilities that the symplectic extensions of unitary algebras provide, and made use of their classification properties in order to achieve generalized descriptions of the nuclear collective behavior.

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