NUCLEAR THEORY, Vol. 30 (2011)
eds. A. Georgieva, N. Minkov, Heron Press, Sofia

# The Deformation Dependent Mass Davidson Model of Atomic Nuclei

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**Abstract.** It is shown how a SUSYQM $\beta$ vibration, which describes the axial deformation of atomic nuclei, leads to a Deformation Dependent Mass behavior for the Davidson potential. Tables which compare theoretical predictions with experimental data are presented for the axially symmetric prolate behavior.

## 1 Introduction

The Deformation Dependent Mass Davidson Model is an extension of the well known Bohr-Mottelson Hamiltonian [1] for the atomic nuclei. It primarily refers to the mass dependence on the deformation and secondary to the Davidson behavior for the potential of the $\beta$-vibration.

The resulting Hamiltonian is solved using techniques of SUSYQM as dictated by [2]. Such a technique is the integrability condition which is called Shape Invariance. A Schroedinger equation is exactly solvable if and only if the potential term is Shape Invariant [3]. The Davidson potential is known for its shape invariant behavior [4]. In the case of a deformation dependent mass in the Bohr Hamiltonian [5], an effective potential is also present. Shape invariance states that its behavior should be that of the Davidson potential. This is the way to determine the functional dependence of mass on the deformation as discussed in [6].

This article will be devoted solely in the solution of the radial equation for the $\beta$ vibrations. The radial equation has a common form for the $\gamma$ unstable and axially symmetric prolate nuclei and the treatment in each case can be found in [7]. Numerical predictions for the spectra are shown for axially symmetric prolate nuclei. Finally a complete comparison for the ground state, $\beta_1$ and $\gamma_1$ bands of $^{162}$Dy and $^{238}$U is presented.

## 2 SUSYQM for the $\beta$ Vibrations

Quesne and Tkachuk begun to study SUSYQM methods for non-pointlike quantum oscillators [8, 9], that is harmonic oscillators with non-pointlike excitations. In principle, in such an oscillator the Heisenberg uncertainty relations are modified and this guides the modification of the canonical commutation relations. In [2] the equivalence of such an oscillator with a Schroedinger equation of a position dependent mass problem was established. In [5] a Schroedinger equation of the Bohr-Mottelson type was presented for the case of a mass dependent on the $\beta$ degree of freedom. Therefore, based on the Quesne and Tkachuk equivalence we construct the phase space of the $\beta$ degree of freedom with commutation relations,

$$[\beta, p_\beta] = i\hbar f(\beta). \tag{1}$$

The function $f(\beta)$ is called the deformation function. Because of its presence, in the second quantization procedure, the ladder operators will not be as usual but modified in generally as,

$$A^\pm \to A^\pm(a,\mu,\nu) = \mp\sqrt{f(a;\beta)}\frac{d}{d\beta}\sqrt{f(a;\beta)} + W(\mu,\nu;\beta). \tag{2}$$

Here a deformed momentum operator is introduced through the deformation function $f(a;\beta)$ and the superpotential $W(\mu,\nu;\beta)$, which signals the SUSYQM method. From the parameters $(a,\mu,\nu)$, only $a$ will remain free. The Hamiltonian corresponding to these ladder operators will of course give good quantum numbers for the stationary states of $\beta$-vibrations, characterized by a function $R(\beta)$.

The principal SUSYQM demand states that the action of the Hamiltonian operator to the ground state shall give zero. The parameter $\varepsilon_0$ is introduced, which is assumed to be the energy of the ground state, and therefore the SUSYQM method is valid for the Hamiltonian,

$$A^+(a,\mu,\nu)A^-(a,\mu,\nu) = H - \varepsilon_0, \tag{3}$$

which gives zero eigenvalue for the vacuum. This Hamiltonian shall correspond to the radial equation which is [7],

$$HR = -\left(\sqrt{f}\frac{d}{d\beta}\sqrt{f}\right)^2 R + 2uR = 2\epsilon R. \tag{4}$$

This correspondence emerges the equation,

$$W^2(\mu,\nu;\beta) - f(\beta)W'(\mu,\nu;\beta) + \varepsilon_0 = 2u(\beta). \tag{5}$$

Now, the main result of [2] is that the Schroedinger equation (4) is also obtained for the case of a position dependent mass problem, as discussed in [5, 10] with a change in the potential,

$$u \to u_{eff} = u + \frac{1}{4}ff'' + \frac{1}{6}(f')^2. \tag{6}$$

Therefore the energy of the ground state $\varepsilon_0$ can be determined from the $u_{eff}$. This can be done if the specific potential $u_{eff}(\beta)$ has known superpotential and deformation function.

## 2.1 Shape Invariance and the Davidson Potential

Schroedinger equation is known to be exactly solvable for the Davidson potential [11]. Shape invariance states that a potential gives exact solutions if and only if retains the same functional dependence under the change of its parameters. In the figure below, shape invariance is shown for the Davidson's parameter $\beta_0$ which is fitted to a specific nucleus, namely the minimum reflects the ground state of the $\beta$ vibrations.

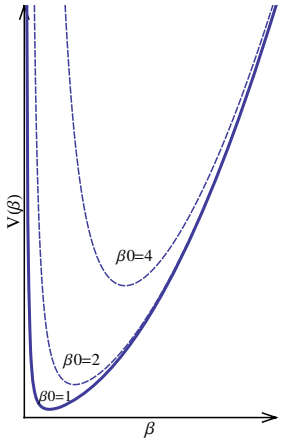


Figure 1. The Davidson Potential $V(\beta) = \beta^2 + \frac{\beta_0^4}{\beta^2}$ and its shape invariant behavior under a parameter shift.

We extend the shape invariance condition for the effective potential. This means that $u_{eff}$ should retain the Davidson behavior for every change in the parameters, with

$$W^2(\mu,\nu;\beta) - f(\beta)W'(\mu,\nu;\beta) + \varepsilon_0 = 2u_{eff}(\beta) = k_1\beta^2 + k_0 + \frac{k_{-1}}{\beta^2}. \quad (7)$$

In [4], classes of shape invariant potentials have been studied with the identification of their corresponding superpotentials and deformation functions. The superpotential and deformation function for the Davidson case are,

$$W(\beta) = \frac{\mu}{\beta} + \nu\beta \qquad , \quad f(\beta) = 1 + a\beta^2. \quad (8)$$

## 3 Energy spectrum

From [5] the coefficients of the effective potential are,

$$k_1 = 2 + a^2(12 + \Lambda), \qquad k_0 = a(13 + 2\Lambda), \qquad k_{-1} = 2 + \Lambda + 2\beta_0^4, \quad (9)$$

with $\Lambda = \tau(\tau + 3)$ for $\gamma$ unstable and $\Lambda = \frac{L(L+1)-K^2}{3} + (6c)(n_\gamma + 1)$ for the axially symmetric behavior. The parameter $c$ controls $\gamma$ stiffness and $n_\gamma$ is the quantum number for $\gamma$ vibrations.

### 3.1 Ground state band

With these equations the energy of the ground state is determined from the expressions,

$$\mu(\mu + 1) = k_{-1}, \quad \nu(\nu - a) = k_1, \quad 2\mu\nu + \mu a - \nu + \varepsilon_0 = k_0, \qquad (10)$$

$$\mu = -\frac{1}{2}\left(1+\sqrt{9+\Lambda+8\beta_0^4}\right), \quad \nu = \frac{a}{2}\left(1+\sqrt{1+\frac{8+4a^2(12+\Lambda)}{a^2}}\right). \quad (11)$$

From these equations the energy of the ground state $\epsilon_0$ is found to be

$$\epsilon_0 = \frac{19}{4}a + \frac{5}{2}a + \frac{1}{2}\sqrt{a^2 + 4k_1} + \frac{a}{2}\sqrt{1 + 4k_{-1}} + \frac{1}{4}\sqrt{(a^2 + 4k_1)(1 + 4k_{-1})} + a\Lambda. \quad (12)$$

Actually this is the energy of the ground state band for the $\beta$ vibration, because of the $\Lambda$ dependence in each case.

### 3.2 $\beta$ Bands

In SUSYQM the operators $A^+(a, \mu, \nu)A^-(a, \mu, \nu)$ and $A^-(a, \mu, \nu)A^+(a, \mu, \nu)$ generate two isospectral Hamiltonians. For the determination of the excited states the extension is not restricted to a pair but to a hierarchy of Hamiltonians

$$H_i = A_i^+ A_i^- + \sum_{j=0}^{i} \varepsilon_j, \qquad i = 0, 1, 2, \ldots, \qquad (13)$$

The shape invariance condition [3] here gives the equation

$$A_i^- A_i^+ = A_{i+1}^+ A_{i+1}^- + \varepsilon_{i+1}, \qquad (14)$$

which means that every excited state of the Hamiltonian under solution is isospectral with the ground state of the $j$-th member of the hierarchy. Therefore in order to obtain the energy of an arbitrary $n$-th excited state, the hierarchy must be terminated to a certain member and we call this member the $n$-th, which is therefore the first quantum number.

Table 1. Comparison of theoretical predictions of the DDM Davidson to experimental data [12] of axially symmetric prolate deformed rare earth and actinide nuclei with $R_{4/2} > 2.9$ and known $0_2^+$ and $2_\gamma^+$ states. The $R_{4/2} = E(4_1^+)/E(2_1^+)$ ratios, as well as the $\beta$ and $\gamma$ bandheads, normalized to the $2_1^+$ state and labelled by $R_{0/2} = E(0_\beta^+)/E(2_1^+)$ and $R_{2/2} = E(2_\gamma^+)/E(2_1^+)$ respectively, are shown. $\beta_0$, $a$, and $c$ are free parameters, related to the Davidson potential, to the dependence of the mass on the deformation, and to the $\gamma$-stiffness. Eq. (21) defines the quality measure $\sigma$, while the angular momenta of the highest levels of the ground state, $\beta$ and $\gamma$ bands included in the rms fit and the total number of levels involved in the fit, for each nucleus can be found in Ref. [7].

| Nucleus | $R_{4/2}$ exp | $R_{4/2}$ th | $R_{0/2}$ exp | $R_{0/2}$ th | $R_{2/2}$ exp | $R_{2/2}$ th | $\beta_0$ | $c$ | $a$ | $\sigma$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $^{150}$Nd | 2.93 | 3.13 | 5.2 | 7.9 | 8.2 | 5.8 | 0.0 | 2.1 | 0.003 | 2.012 |
| $^{152}$Sm | 3.01 | 3.14 | 5.6 | 8.4 | 8.9 | 6.5 | 0.0 | 2.4 | 0.000 | 3.327 |
| $^{154}$Sm | 3.25 | 3.27 | 13.4 | 13.0 | 17.6 | 18.6 | 1.30 | 6.9 | 0.021 | 0.515 |
| $^{154}$Gd | 3.02 | 3.09 | 5.5 | 6.5 | 8.1 | 4.1 | 0.0 | 1.4 | 0.024 | 3.546 |
| $^{156}$Gd | 3.24 | 3.25 | 11.8 | 10.8 | 13.0 | 14.3 | 0.0 | 5.3 | 0.026 | 0.933 |
| $^{158}$Gd | 3.29 | 3.29 | 15.0 | 14.5 | 14.9 | 15.1 | 1.99 | 5.3 | 0.025 | 0.323 |
| $^{160}$Gd | 3.30 | 3.30 | 17.6 | 17.3 | 13.1 | 13.2 | 2.38 | 4.5 | 0.020 | 0.125 |
| $^{162}$Gd | 3.29 | 3.30 | 19.8 | 19.8 | 12.0 | 12.1 | 2.52 | 4.1 | 0.008 | 0.078 |
| $^{156}$Dy | 2.93 | 3.13 | 4.9 | 7.4 | 6.5 | 5.3 | 0.0 | 1.9 | 0.014 | 1.789 |
| $^{158}$Dy | 3.21 | 3.22 | 10.0 | 9.6 | 9.6 | 10.3 | 0.26 | 3.8 | 0.023 | 0.496 |
| $^{160}$Dy | 3.27 | 3.27 | 14.7 | 14.7 | 11.1 | 12.1 | 1.92 | 4.3 | 0.005 | 0.510 |
| $^{162}$Dy | 3.29 | 3.30 | 17.3 | 15.7 | 11.0 | 11.2 | 2.23 | 3.8 | 0.020 | 0.742 |
| $^{164}$Dy | 3.30 | 3.30 | 22.6 | 22.5 | 10.4 | 10.2 | 2.68 | 3.4 | 0.000 | 0.100 |
| $^{166}$Dy | 3.31 | 3.31 | 15.0 | 14.9 | 11.2 | 11.2 | 2.39 | 3.7 | 0.047 | 0.077 |
| $^{160}$Er | 3.10 | 3.16 | 7.1 | 8.1 | 6.8 | 6.6 | 0.00 | 2.4 | 0.013 | 0.699 |
| $^{162}$Er | 3.23 | 3.23 | 10.7 | 10.7 | 8.8 | 10.1 | 1.29 | 3.7 | 0.013 | 0.770 |
| $^{164}$Er | 3.28 | 3.27 | 13.6 | 12.2 | 9.4 | 9.6 | 1.83 | 3.3 | 0.026 | 0.918 |
| $^{166}$Er | 3.29 | 3.28 | 18.1 | 16.8 | 9.8 | 9.9 | 2.22 | 3.4 | 0.002 | 0.698 |
| $^{168}$Er | 3.31 | 3.31 | 15.3 | 14.4 | 10.3 | 10.2 | 2.29 | 3.4 | 0.041 | 0.404 |
| $^{170}$Er | 3.31 | 3.30 | 11.3 | 10.1 | 11.9 | 12.9 | 1.64 | 4.4 | 0.083 | 0.837 |
| $^{162}$Yb | 2.92 | 3.07 | 3.6 | 6.8 | 4.8 | 4.0 | 0.00 | 1.4 | 0.003 | 1.036 |
| $^{164}$Yb | 3.13 | 3.18 | 7.9 | 8.3 | 7.0 | 7.4 | 0.00 | 2.7 | 0.023 | 0.357 |
| $^{166}$Yb | 3.23 | 3.23 | 10.2 | 8.9 | 9.1 | 9.7 | 0.66 | 3.5 | 0.038 | 0.973 |
| $^{168}$Yb | 3.27 | 3.26 | 13.2 | 11.2 | 11.2 | 11.5 | 1.52 | 4.1 | 0.028 | 1.070 |
| $^{170}$Yb | 3.29 | 3.27 | 12.7 | 11.2 | 13.6 | 14.1 | 1.36 | 5.1 | 0.035 | 0.963 |
| $^{172}$Yb | 3.31 | 3.30 | 13.2 | 12.2 | 18.6 | 18.9 | 1.66 | 6.6 | 0.055 | 0.742 |
| $^{174}$Yb | 3.31 | 3.31 | 19.4 | 19.3 | 21.4 | 21.5 | 2.44 | 7.5 | 0.019 | 0.104 |
| $^{176}$Yb | 3.31 | 3.30 | 13.9 | 13.7 | 15.4 | 15.5 | 1.97 | 5.4 | 0.036 | 0.287 |
| $^{178}$Yb | 3.31 | 3.27 | 15.7 | 15.5 | 14.5 | 14.6 | 1.88 | 5.3 | 0.000 | 0.127 |
| $^{166}$Hf | 2.97 | 3.08 | 4.4 | 6.9 | 5.1 | 4.3 | 0.00 | 1.5 | 0.006 | 0.873 |
| $^{168}$Hf | 3.11 | 3.17 | 7.6 | 8.1 | 7.1 | 6.9 | 0.00 | 2.5 | 0.023 | 0.494 |
| $^{170}$Hf | 3.19 | 3.21 | 8.7 | 8.7 | 9.5 | 8.8 | 0.00 | 3.2 | 0.033 | 0.970 |
| $^{172}$Hf | 3.25 | 3.24 | 9.2 | 9.8 | 11.3 | 11.7 | 0.00 | 4.3 | 0.031 | 0.549 |

Table 1. (continued)

| Nucleus | $R_{4/2}$ exp | $R_{4/2}$ th | $R_{0/2}$ exp | $R_{0/2}$ th | $R_{2/2}$ exp | $R_{2/2}$ th | $\beta_0$ | $c$ | $a$ | $\sigma$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $^{174}$Hf | 3.27 | 3.25 | 9.1 | 10.4 | 13.5 | 13.6 | 0.00 | 5.0 | 0.033 | 0.832 |
| $^{176}$Hf | 3.28 | 3.28 | 13.0 | 11.5 | 15.2 | 16.1 | 1.31 | 5.8 | 0.038 | 0.950 |
| $^{178}$Hf | 3.29 | 3.28 | 12.9 | 12.3 | 12.6 | 13.0 | 1.70 | 4.6 | 0.028 | 0.356 |
| $^{180}$Hf | 3.31 | 3.30 | 11.8 | 11.5 | 12.9 | 13.0 | 1.92 | 4.4 | 0.068 | 0.157 |
| $^{176}$W | 3.22 | 3.21 | 7.8 | 9.1 | 9.6 | 9.5 | 0.00 | 3.5 | 0.027 | 0.881 |
| $^{178}$W | 3.24 | 3.22 | 9.4 | 8.6 | 10.5 | 8.9 | 0.00 | 3.2 | 0.039 | 0.987 |
| $^{180}$W | 3.26 | 3.25 | 14.6 | 13.1 | 10.8 | 11.5 | 1.64 | 4.2 | 0.000 | 0.603 |
| $^{182}$W | 3.29 | 3.29 | 11.3 | 11.5 | 12.2 | 12.5 | 1.77 | 4.3 | 0.050 | 0.195 |
| $^{184}$W | 3.27 | 3.28 | 9.0 | 8.9 | 8.1 | 8.0 | 1.57 | 2.7 | 0.080 | 0.093 |
| $^{186}$W | 3.23 | 3.25 | 7.2 | 7.2 | 6.0 | 6.3 | 1.20 | 2.1 | 0.099 | 0.130 |
| $^{176}$Os | 2.93 | 3.10 | 4.5 | 6.9 | 6.4 | 4.6 | 0.00 | 1.6 | 0.016 | 1.747 |
| $^{178}$Os | 3.02 | 3.12 | 4.9 | 7.2 | 6.6 | 5.1 | 0.00 | 1.8 | 0.017 | 1.836 |
| $^{180}$Os | 3.09 | 3.22 | 5.6 | 7.1 | 6.6 | 6.9 | 0.00 | 2.4 | 0.078 | 1.021 |
| $^{184}$Os | 3.20 | 3.21 | 8.7 | 9.9 | 7.9 | 8.5 | 1.21 | 3.1 | 0.011 | 0.886 |
| $^{186}$Os | 3.17 | 3.19 | 7.7 | 7.0 | 5.6 | 6.0 | 0.00 | 2.1 | 0.063 | 0.702 |
| $^{188}$Os | 3.08 | 3.15 | 7.0 | 7.2 | 4.1 | 4.4 | 1.07 | 1.5 | 0.033 | 0.170 |
| $^{190}$Os | 2.93 | 3.07 | 4.9 | 5.6 | 3.0 | 3.1 | 0.00 | 1.0 | 0.051 | 0.419 |
| $^{228}$Ra | 3.21 | 3.24 | 11.3 | 11.0 | 13.3 | 13.3 | 0.57 | 5.0 | 0.016 | 0.177 |
| $^{228}$Th | 3.24 | 3.26 | 14.4 | 14.3 | 16.8 | 17.0 | 1.50 | 6.4 | 0.002 | 0.214 |
| $^{230}$Th | 3.27 | 3.27 | 11.9 | 11.6 | 14.7 | 14.7 | 1.44 | 5.3 | 0.034 | 0.243 |
| $^{232}$Th | 3.28 | 3.28 | 14.8 | 14.0 | 15.9 | 16.5 | 1.80 | 5.9 | 0.022 | 0.426 |
| $^{232}$U | 3.29 | 3.29 | 14.5 | 13.8 | 18.2 | 18.4 | 1.74 | 6.6 | 0.028 | 0.394 |
| $^{234}$U | 3.30 | 3.30 | 18.6 | 18.3 | 21.3 | 21.8 | 2.19 | 7.8 | 0.011 | 0.244 |
| $^{236}$U | 3.30 | 3.30 | 20.3 | 20.0 | 21.2 | 21.2 | 2.38 | 7.5 | 0.009 | 0.143 |
| $^{238}$U | 3.30 | 3.31 | 20.6 | 20.6 | 23.6 | 24.7 | 2.38 | 8.8 | 0.009 | 0.665 |
| $^{238}$Pu | 3.31 | 3.31 | 21.4 | 21.4 | 23.3 | 23.3 | 2.61 | 8.1 | 0.016 | 0.067 |
| $^{240}$Pu | 3.31 | 3.31 | 20.1 | 19.9 | 26.6 | 26.6 | 2.40 | 9.4 | 0.018 | 0.117 |
| $^{242}$Pu | 3.31 | 3.31 | 21.5 | 21.4 | 24.7 | 24.7 | 2.52 | 8.7 | 0.012 | 0.107 |
| $^{248}$Cm | 3.31 | 3.31 | 25.0 | 24.8 | 24.2 | 24.3 | 2.72 | 8.5 | 0.004 | 0.159 |
| $^{250}$Cf | 3.32 | 3.31 | 27.0 | 26.9 | 24.2 | 24.2 | 2.88 | 8.4 | 0.003 | 0.053 |

In terms of the superpotential and the deformation function, the shape invariance condition gives,

$$W^2(\mu_i,\nu_i;\beta)+f(\beta)W'(\mu_i,\nu_i;\beta) = W^2(\mu_{i+1},\nu_{i+1};\beta)-f(\beta)W'(\mu_{i+1},\nu_{i+1};\beta)+\varepsilon_{i+1}, \quad (15)$$

where $i=0,1,2,\ldots$, $\mu_0=\mu$, $\nu_0=\nu$, and is equivalent to

$$\Big(\frac{\mu_i}{\beta}+\nu_i\beta\Big)^2+(1+a\beta^2)\Big(-\frac{\mu_i}{\beta^2}+\nu_i\Big) = \Big(\frac{\mu_{i+1}}{\beta}+\nu_{i+1}\beta\Big)^2-(1+a\beta^2)\Big(-\frac{\mu_{i+1}}{\beta^2}+\nu_{i+1}\Big)+\varepsilon_{i+1}. \quad (16)$$

We obtain,

$$\begin{aligned}
&\mu_i(\mu_i-1)=\mu_{i+1}(\mu_{i+1}+1),\\
&\nu_i(\nu_i+a)=\nu_{i+1}(\nu_{i+1}-a),\\
&2\mu_i\nu_i-\mu_i a+\nu_i=2\mu_{i+1}\nu_{i+1}+\mu_{i+1}a-\nu_{i+1}+\varepsilon_{i+1}.
\end{aligned} \tag{17}$$

Their solutions are

$$\mu_i=\mu-i, \qquad \nu_i=\nu+ia, \tag{18}$$

and

$$\varepsilon_{i+1}=2(\mu_i\nu_i-\mu_{i+1}\nu_{i+1})-(\mu_i+\mu_{i+1})a+\nu_i+\nu_{i+1}. \tag{19}$$

From these relations and the termination to the $n$-th member the energy spectrum of the model is determined as,

$$\begin{aligned}
\epsilon_n=\tfrac{1}{2}\sum_{i=0}^{n}\varepsilon_i=\tfrac{1}{2}[k_0+\tfrac{1}{2}a(3+2\Delta_1+2\Delta_2+\Delta_1\Delta_2)\\
+2a(2+\Delta_1+\Delta_2)n+4an^2], \qquad n=0,1,2,\ldots,
\end{aligned} \tag{20}$$

with $\Delta_1\equiv\sqrt{1+4k_{-1}}$, $\Delta_2\equiv\sqrt{1+4\frac{k_1}{a^2}}$. The ground state band is obtained from $n=0$, while the quasi-$\beta_1$ band is obtained from $n=1$, and the quasi-$\beta_2$ band is obtained from $n=2$.

## 4 Fitting

In [7] the above energy spectrum was fitted for the cases of $\gamma$-unstable and deformed axially symmetric nuclei. The fitting measure was the Gaussian error,

$$\sigma=\sqrt{\frac{\sum_{i=1}^{n}(E_i(exp)-E_i(th))^2}{(n-1)E(2_1^+)^2}}. \tag{21}$$

In the case of deformed axially symmetric nuclei, three free parameters remain for the comparison to the experimental data. These are $\beta_0$, which is fitted to a specific nucleus and shows the deformation of its ground state, $a$, which shows the DDM magnitude in this nucleus, and $c$, which controls the $\gamma$-vibrations for the axially prolate case. Some results are shown in two tables.

In the first table the agreement for the spectra is very good in most cases. Disagreement is observed in those nuclei which are close to the X(5) limit, such as $^{150}$Nd, $^{152}$Sm, $^{154}$Gd, $^{156}$Dy. This behavior is expected, because a flat potential is appropriate for the X(5) limit, not the Davidson one.

In the second table the ground state, $\beta_1$ and $\gamma_1$ bands in the cases of $^{238}$U and $^{162}$Dy are compared to experimental data [12]. Both the bandheads and the spacings within bands are in general well reproduced. This is particularly true for the ground state and the $\gamma_1$ bands. The deviation in the gsb of $^{162}$Dy reaches

Table 2. Normalized [to the energy of the first excited state, $E(2_1^+)$] energy levels of the ground state band (gsb) and the $\beta_1$ and $\gamma_1$ bands of $^{162}$Dy and $^{238}$U, obtained from the Bohr Hamiltonian with $\beta$-dependent mass for axially symmetric prolate deformed nuclei using the parameters given in Table 1, compared to available experimental data [12].

| | $^{162}$Dy | $^{162}$Dy | $^{238}$U | $^{238}$U | | $^{162}$Dy | $^{162}$Dy | $^{238}$U | $^{238}$U |
|---|---|---|---|---|---|---|---|---|---|
| L | exp | th | exp | th | L | exp | th | exp | th |
| | gsb | gsb | gsb | gsb | | $\gamma_1$ | $\gamma_1$ | $\gamma_1$ | $\gamma_1$ |
| 0 | 0.00 | 0.00 | 0.00 | 0.00 | 2 | 11.0 | 11.2 | 23.6 | 24.7 |
| 2 | 1.00 | 1.00 | 1.00 | 1.00 | 3 | 11.9 | 12.1 | 24.6 | 25.5 |
| 4 | 3.29 | 3.30 | 3.30 | 3.31 | 4 | 13.2 | 13.3 | 25.9 | 26.7 |
| 6 | 6.80 | 6.80 | 6.84 | 6.86 | 5 | 14.7 | 14.7 | 27.4 | 28.1 |
| 8 | 11.41 | 11.41 | 11.54 | 11.57 | 6 | 16.4 | 16.5 | 29.2 | 29.8 |
| 10 | 17.04 | 17.01 | 17.27 | 17.33 | 7 | 18.5 | 18.5 | 31.2 | 31.7 |
| 12 | 23.57 | 23.49 | 23.97 | 24.06 | 8 | 20.7 | 20.8 | 33.5 | 33.9 |
| 14 | 30.90 | 30.74 | 31.51 | 31.63 | 9 | 23.3 | 23.3 | 36.0 | 36.3 |
| 16 | 38.90 | 38.70 | 39.82 | 39.97 | 10 | 25.9 | 26.0 | 38.8 | 39.0 |
| 18 | 47.58 | 47.28 | 48.78 | 48.98 | 11 | 29.0 | 28.9 | 41.7 | 41.9 |
| 20 | | | 58.31 | 58.61 | 12 | 31.4 | 32.1 | 44.9 | 45.0 |
| 22 | | | 68.31 | 68.77 | 13 | 35.5 | 35.5 | 48.3 | 48.3 |
| 24 | | | 78.71 | 79.44 | 14 | 39.4 | 39.9 | 51.9 | 51.8 |
| 26 | | | 89.46 | 90.55 | 15 | | | 55.7 | 55.5 |
| 28 | | | 100.57 | 102.08 | 16 | | | 59.7 | 59.4 |
| 30 | | | 112.10 | 113.99 | 17 | | | 63.9 | 63.4 |
| | | | | | 18 | | | 68.2 | 67.7 |
| | $\beta_1$ | $\beta_1$ | $\beta_1$ | $\beta_1$ | 19 | | | 72.7 | 72.0 |
| 0 | 17.3 | 15.7 | 20.6 | 20.6 | 20 | | | 77.3 | 76.6 |
| 2 | 18.0 | 16.7 | 21.5 | 21.6 | 21 | | | 82.1 | 81.3 |
| 4 | 19.5 | 19.0 | 23.5 | 24.0 | 22 | | | 87.0 | 86.1 |
| 6 | 21.9 | 22.6 | | | 23 | | | 91.9 | 91.0 |
| 8 | 24.6 | 27.4 | | | 24 | | | 97.0 | 96.1 |
| | | | | | 25 | | | 102.1 | 101.3 |
| | | | | | 26 | | | 107.4 | 106.6 |
| | | | | | 27 | | | 112.7 | 112.0 |

0.6% at $L = 18$, while in the gsb of $^{238}$U it reaches 1.7% at $L = 30$. The experimental levels of the $\gamma_1$ band of $^{162}$Dy (up to $L = 14$) extend over 28.4 energy units, while the corresponding theoretical predictions spread over 28.7 units, the difference being of the order of 1%. Similarly in $^{238}$U the experimental spread of the $\gamma_1$ band (up to $L = 27$) is 89.1 energy units, while the theoretical one is 87.3 units, the difference being of the order of 2%.

On the other hand, the theoretical level spacings within the $\beta_1$ bands are larger than the experimental ones. It is worth mentioning that results for $B(E2)$ transition rates in the two cases [7], reveal a similar behavior. B(E2)'s are in overall good agreement with the experimental data, apart from the $\beta_1 \rightarrow$ gsb

transitions. The theoretical predictions are overestimating these transitions.

This behavior along with the $\beta_1$ bands could be seen as an effect of the Davidson potential. It rises too fast in its right part as seen in Figure 1, thus producing a large gap between the ground state and the $\beta_1$ bands and in addition increasing their interlevel spacing. However, as it has been pointed out in [13], the form of the quadrupole operator should be tested by changing the mass coefficients of the Bohr Hamiltonian.

## 5 Conclusion

The motivation to adopt a DDM framework in the Bohr Hamiltonian is described in [6]. A treatment of the $\beta$ vibrations through SUSYQM techniques results immediately in a DDM behavior. The application of the Davidson potential in the resulting Schroedinger equation yields an overall good agreement for the axially symmetric prolate nuclei, apart from X(5) candidates. An encouraging fact for the model is the agreement to the data for large values of angular momenta in the ground state and $\gamma_1$ bands for the cases of $^{238}$U and $^{162}$Dy. However, the behavior of $\beta_1$ bands does not share this success.

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