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Approximate SU(3) Symmetry Inside the Symmetry Triangle of the Interacting Boson Model

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Abstract. The U(5), SU(3), and O(6) symmetries of the Interacting Boson Model (IBM) have been traditionally placed at the vertices of the symmetry triangle, while an O(5) symmetry is known to hold along the U(5)-O(6) side of the triangle. We construct for the first time a symmetry line in the interior of the triangle, along which the SU(3) symmetry is preserved. This is achieved by using the contraction of the SU(3) algebra to the algebra of the rigid rotator in the large boson number limit of the IBM. The line extends from the SU(3) vertex to near the critical line of the first order shape/phase transition separating the spherical and prolate deformed phases. It lies within the Alhassid–Whelan arc of regularity, the unique valley of regularity connecting the SU(3) and U(5) vertices amidst chaotic regions, thus providing an explanation for its existence.

1 Introduction

The study, carried out by Alhassid and Whelan, of chaotic properties at the symmetry triangle [1] of the IBM [2], brought to the surface a region [3–5] of nearly regular behavior, inside the symmetry triangle, connecting the U(5) and SU(3) vertices, called the Alhassid-Whelan arc of regularity. Another regular region was known to exist along the U(5)–O(6) leg of the triangle. While the existence of the latter is known to be due to the underlying SO(5) symmetry, a common subalgebra of both U(5) and O(6), the symmetry underlying the arc has remained an open question.

The presence of (near) regularity presupposes the existence of some underlying (approximate) symmetry. A well–known hallmark of SU(3) symmetry, is the degeneracy within sets of bands comprising a given irreducible representation (irrep), such as those between the levels of the β band and those (with the same L) of the γ band. At a recent study [6], by imposing the $2^+_\gamma = 2^+_\beta$ degeneracy in the IBM framework, a line was found inside the symmetry triangle of the IBM, starting from the SU(3) vertex, reaching the shape/phase coexistence region [7], lying very close to the arc of regularity. The analysis of Ref. [6] was limited to the low-lying part of the spectrum, while the study of Alhassid and Whelan [3–5] took into consideration the whole spectrum.

In the present work, we study the presence of regularity in the arc using quantal signatures of chaos and we study Hamiltonians that approximately commute with the SU(3) generators by considering the large–boson–number limit of the IBA. We also derive an analytic expression for the locus of points of these Hamiltonians inside the symmetry triangle of the IBM. This is achieved by using the contraction [8] of the SU(3) algebra to the [R^5]SO(3) algebra [9, 10] of the rigid rotator [11].

In Section 2, we describe the Hamiltonian of the model. In Section 3, using statistical tools, we see whether the arc is characterized by regularity or chaos. In Section 4, the analytic expressions of the locus of points of the commutation of the Hamiltonian with the SU(3) generators are derived, using two different methods. The conclusions are presented in Section 5. Necessary values of the commutation relations and matrix elements are given in the Appendices of [12].

2 The Hamiltonian

The Interacting Boson Model (IBM) comprises a system of N two-state bosons: a lower energy L=0 (s-boson) state and a higher energy L=2 (d-boson) state. The 36 bilinear combinations $(s^\dagger s, s^\dagger \tilde{d}_\mu, d^\dagger_\mu s, d^\dagger_\mu \tilde{d}_\nu)$ form a U(6) dynamical algebra. The model has 3 dynamical symmetries, U(5), SU(3), and O(6), which correspond to vibrational, rotational and γ-unstable nuclei respectively. These dynamical symmetries are placed at the vertices of the symmetry triangle (Casten's triangle) [1], shown in Figure 1, which is the parameter space of the model.

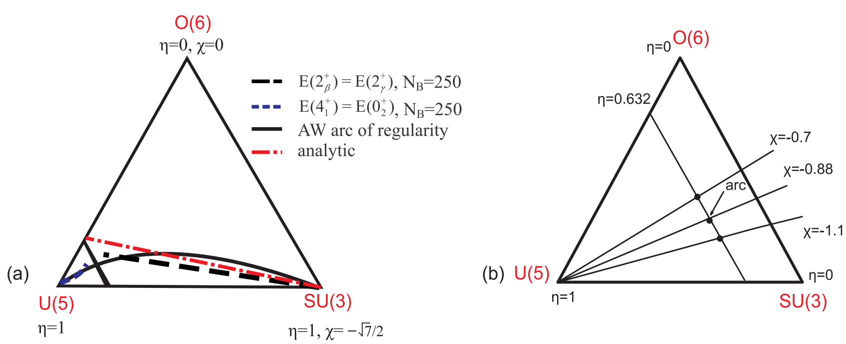

Figure 1. (a) IBA symmetry triangle with the three dynamical symmetries, the Alhassid–Whelan arc of regularity (10), the present line of (9) (labelled as analytic), the loci of the degeneracies $E(2^+_\beta) = E(2^+_\gamma)$ (dashed line on the right for $N_B = 250$ [6] and $E(4^+_1) = E(0^+_2)$ (doted line on the left for $N_B = 250$ [6]). (b) The four different points where calculations were performed.

In what follows we use the IBM Hamiltonian [3–5]

$$H(\eta, \chi) = c\left[\eta \hat{n}_d + \frac{\eta - 1}{N_B}\hat{Q}_\chi \hat{Q}_\chi\right], \tag{1}$$

where $\hat{n}_d = d^\dagger \cdot \tilde{d}$ is the d boson number operator, $\hat{Q}_\chi = (s^\dagger \tilde{d} + d^\dagger s) + \chi(d^\dagger \tilde{d})^{(2)}$ is the quadrupole operator, and N_B is the number of valence bosons. The parameters (η, χ) are the coordinates of the triangle and serve for symmetry breaking. η ranges from 0 to 1, and χ ranges from 0 to $\frac{-\sqrt{7}}{2} = -1.32$. Numerical calculations of energy levels have been performed using the code IBAR [13] which can handle bosons up to N_B=250.

3 Quantal signatures of chaos

As was already mentioned, the study of Ref. [6] was limited to the low part of the spectrum, where degeneracies occur. In order to study the whole spectrum one needs to use statistical tools. One of them is the nearest neighbor spacing distribution, P(S). It is the probability that two adjacent energies differ by an amount of S.

In this section, using the nearest neighbor spacing distribution, we compare, in terms of chaoticity, a point which is characterized by regularity, like the point on the SU(3) vertex, which has SU(3) symmetry, a point on the arc and two points above and below the arc (Figure 1b). The energies for the whole spectrum at the above four points were found and the nearest spacing distribution of these energies was drawn for each different point. Then, these distributions were fitted to the Brody distribution [14]. The Brody distribution, which is characterized by the parameter ω,

$$P(S) = (1 + \omega) R S^\omega \exp(-R S^{1+\omega}), \tag{2}$$

where $R = \Gamma\left[\frac{2+\omega}{1+\omega}\right]^{1+\omega}$, interpolates between Poisson statistics ($\omega = 0$), which characterize a regular system, and the Wigner distribution ($\omega = 1$), which corresponds to a chaotic system. The value of the ω derived by the fit (Figure 2) implies that indeed the point on the arc is more regular than the points above and below the arc.

4 The SU(3) symmetry

4.1 Commutation relations

In order to see whether a system has an underlying SU(3) symmetry, the Hamiltonian of the system has to commute with the generators of SU(3). The generators of the SU(3) algebra [2] are the angular momentum operators

$$\hat{L}_\xi = \sqrt{10}(d^\dagger \tilde{d})^1_\xi, \tag{3}$$

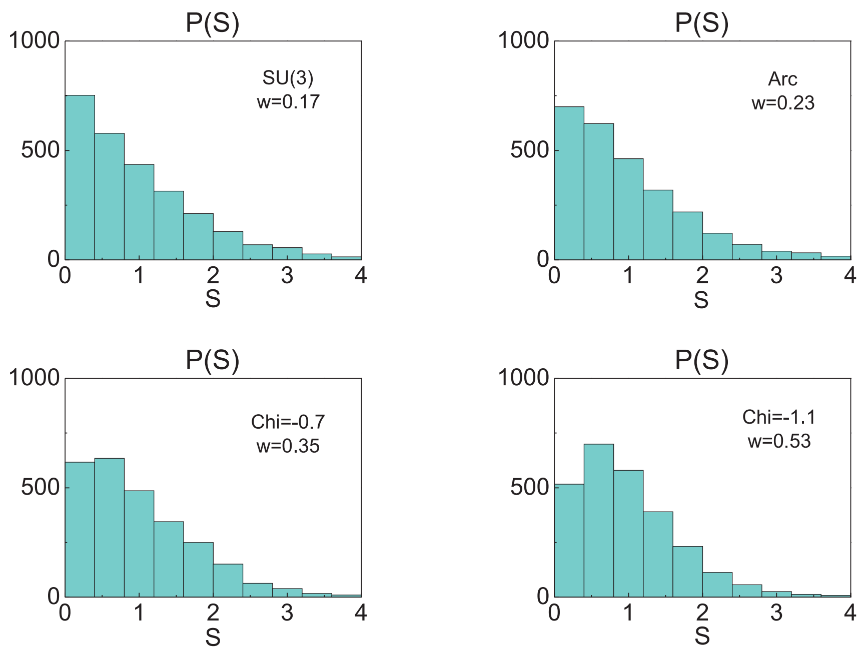

Figure 2. The histograms for the whole spectrum, for the 4 different points described in the text, for L=0 and $N_B = 175$ bosons. The ω values are also shown for each particular point.

and the quadrupole operators

$$\hat{Q}^{(2)}_{SU(3),\xi} = (s^\dagger \tilde{d} + d^\dagger s)^{(2)}_\xi - \frac{\sqrt{7}}{2}(d^\dagger \tilde{d})^{(2)}_\xi . \tag{4}$$

The Hamiltonian of (1) does commute with the angular momentum operators $\hat{L}_\xi$ by construction, since it is a scalar quantity. We will examine the special conditions under which the Hamiltonian also commutes (approximately) with the quadrupole operators.

The first term of the Hamiltonian gives

$$[\hat{H}_1, \hat{Q}^{(2)}_{SU(3),\nu}] = c\eta[\hat{n}_d, \hat{Q}^{(2)}_{SU(3),\nu}] = c\eta(d^\dagger s - s^\dagger \tilde{d})^{(2)}_\nu . \tag{5}$$

Using

$$\hat{Q}^{(2)}_{\chi,\xi} = \hat{Q}^{(2)}_{SU(3),\xi} + \left(\chi + \frac{\sqrt{7}}{2}\right)(d^\dagger \tilde{d})^{(2)}_\xi , \tag{6}$$

in the second term of the Hamiltonian one gets the intermediate result

$$[\hat{Q}^{(2)}_{\chi}\cdot\hat{Q}^{(2)}_{\chi},\hat{Q}^{(2)}_{SU(3),\nu}]=$$
$$=\sum_{\xi}(-1)^{\xi}\Big\{[\hat{Q}^{(2)}_{SU(3),\xi},\hat{Q}^{(2)}_{SU(3),\nu}]\hat{Q}^{(2)}_{\chi,-\xi}+\hat{Q}^{(2)}_{\chi,\xi}[\hat{Q}^{(2)}_{SU(3),-\xi},\hat{Q}^{(2)}_{SU(3),\nu}]$$
$$+\big(\chi+\frac{\sqrt{7}}{2}\big)\big\{[(d^{\dagger}\tilde{d})^{(2)}_{\xi},\hat{Q}^{(2)}_{SU(3),\nu}]\hat{Q}^{(2)}_{\chi,-\xi}+\hat{Q}^{(2)}_{\chi,\xi}[(d^{\dagger}\tilde{d})^{(2)}_{-\xi},\hat{Q}^{(2)}_{SU(3),\nu}]\big\}\Big\}. \quad (7)$$

In the large N limit, (7) can be simplified. In this limit, the eigenvalue expression for the second order Casimir of SU(3) reduces to just the λ^2 term for SU(3) irreducible representations (irreps) (λ,μ) with $\lambda >> \mu$ and hence the ground state band [which belongs to the (2N,0) irrep] becomes energetically isolated from all other excitations. That is, SU(3) effectively reduces to a simple rigid rotator. This situation is formally known as the contraction of SU(3) to R^5[SO(3)] [9, 10] and occurs when the $Q^{(2)}_{SU(3)}$ operators can be replaced by mutually commuting quantities. In the large N limit, where contraction occurs, the commutators in the first two terms in (7) will vanish and terms containing $(d^{\dagger}\tilde{d})^{(k)}$ can be omitted. Furthermore, $\hat{Q}^{(2)}_{\chi}$ can be replaced by $\hat{Q}^{(2)}_{SU(3)}$ and $\hat{Q}^{(2)}_{SU(3)}$ can be replaced by the intrinsic quadrupole moment (a scalar), which is $N\sqrt{2}$ in the present case. So, in the large N limit, the commutator for the second part of the Hamiltonian becomes

$$[\hat{H}_2,\hat{Q}^{(2)}_{SU(3),\nu}]=c(\eta-1)2\sqrt{2}\left(\chi+\frac{\sqrt{7}}{2}\right)(d^{\dagger}s-s^{\dagger}\tilde{d})^{(2)}_{\nu}. \quad (8)$$

However, we want a vanishing commutator, so the coefficients of $(d^{\dagger}s-s^{\dagger}\tilde{d})^{(2)}_{\nu}$ in (7) and (8) should cancel, leading in the large N limit to the condition

$$\chi(\eta)=\frac{1}{2\sqrt{2}}\frac{\eta}{(1-\eta)}-\frac{\sqrt{7}}{2}. \quad (9)$$

The expression of the Alhassid–Whelan arc of regularity [3–5] is given by [15]

$$\chi(\eta)=\frac{\sqrt{7}-1}{2}\eta-\frac{\sqrt{7}}{2}. \quad (10)$$

The expressions of Eqs. (9) and (10) are visualized in Figure 3. The two equations give very similar predictions for values of η between 0 and 0.6, i.e., from the SU(3) vertex until quite close to the critical line.

4.2 Matrix Elements

At the previous subsection, we replaced the quadrupole operator $\hat{Q}^{(2)}_{SU(3)}$ by the intrinsic quadrupole moment. To justify this, we examine the conditions under which the matrix elements of $[\hat{H}_1,\hat{Q}^{(2)}_{SU(3),\nu}]$ and $[\hat{H}_2,\hat{Q}^{(2)}_{SU(3),\nu}]$ lead to a vanishing result, within the ground state band. The consideration of matrix elements within the ground state band is similar to the condition discussed earlier in the context of Eq. (7), where the $Q^{(2)}_{SU(3)}$ operators simplify when the ground state band becomes energetically isolated.

Using the standard formalism for treating matrix elements, one finds for the first and second terms of the Hamiltonian, after replacing $\hat{Q}^{(2)}_{\chi}$ by $\hat{Q}^{(2)}_{SU(3)}$

$$\langle [N],(2N,0),\tilde{\chi}=0,L||[\hat{H}_1,\hat{Q}^{(2)}_{SU(3)}]||[N],(2N,0),\tilde{\chi}=0,L\rangle = -\frac{2}{3\sqrt{7}}c\eta N R_1\sqrt{2L+1}, \tag{11}$$

$$\langle [N],(2N,0),\tilde{\chi}=0,L||[\hat{H}_2,\hat{Q}^{(2)}_{SU(3)}]||[N],(2N,0),\tilde{\chi}=0,L\rangle = \frac{1}{7\sqrt{2}}c(1-\eta)\left(\chi+\frac{\sqrt{7}}{2}\right)q_0 R_2\sqrt{2L+1}, \tag{12}$$

where $\tilde{\chi}$ is the Vergados quantum number [16] and R_1, R_2 are complicated expressions [12] of the number of bosons N and the angular momentum L. In the large N limit and for L not too small, we obtain $R_1 = 1$ and $R_2 = 16/3$. Replacing the intrinsic quadrupole moment q_0 by its value ($N\sqrt{2}$), we get vanishing matrix elements in the large N limit when

$$\chi(\eta) = \frac{\sqrt{7}}{8}\frac{\eta}{(1-\eta)} - \frac{\sqrt{7}}{2}. \tag{13}$$

The expression of (13) is also visualized in Figure 3, where one can remark its similarity to (9).

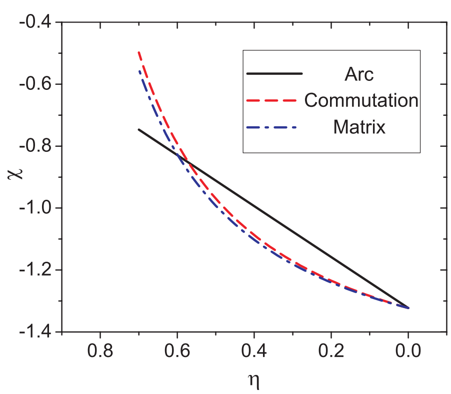

Figure 3. Location of the arc of regularity, as described by the original (10), and as predicted by the findings of the present work, (9) and (13). The η axis has been reversed, in order to correspond directly to Figure 1.

5 Conclusion

The arc of regularity is a narrow strip of nearly regular behavior inside the symmetry triangle of the IBM. The presence of near regularity requires the presence of some underlying approximate symmetry. We have studied the commutation of the IBA Hamiltonian with the SU(3) generators by considering the large–boson–number limit of the IBA and we have determined analytically a line of approximate SU(3) symmetry taking advantage of the contraction of the SU(3) to the R^5[SO(3)], the algebra of the rigid rotator. This line follows closely the Alhassid– Whelan arc of regularity, giving thus an explanation for its existence.

References

[1] R.F. Casten, *Nuclear Structure from a Simple Perspective*, Oxford University Press, Oxford (1990).
[2] F. Iachello, A. Arima, *The Interacting Boson Model*, Cambridge University Press, Cambridge (1987).
[3] Y. Alhassid, N. Whelan, *Phys. Rev. Lett.* **67** (1991) 816.
[4] N. Whelan, Y. Alhassid, *Nucl. Phys. A* **556** (1993) 42.
[5] N. Whelan, Ph. D. thesis, Yale University (1993).
[6] D. Bonatsos, E.A. McCutchan, R.F. Casten, *Phys. Rev. Lett.* **104** (2010) 022502.
[7] F. Iachello, N.V. Zamfir, R.F. Casten, *Phys. Rev. Lett.* **81** (1998) 1191.
[8] E. İnőnű, E.P. Wigner, *Proc. Natl. Acad. Sci.* (N.Y.) **39** (1953) 510.
[9] R. Le Blanc, J. Carvalho, D.J. Rowe, *Phys. Lett. B* **140** (1984) 155.
[10] D.J. Rowe, *Prog. Part. Nucl. Phys.* **37** (1996) 265.
[11] H. Ui, *Prog. Theor. Phys.* **44** (1970) 153.
[12] D. Bonatsos, S. Karampagia, R.F. Casten, *Phys. Rev. C* **53** (2011) 054313.
[13] R.J. Casperson, Ph.D. thesis, Yale University (2010).
[14] T.A. Brody, J. Flores, J.B. French, P.A. Mello, A. Pandey, S.S. M. Wong, *Rev. Mod. Phys.* **53** (1981) 3.
[15] P. Cejnar, J. Jolie, *Phys. Rev. E* **58** (1998) 387.
[16] J.D. Vergados, *Nucl. Phys. A* **111** (1968) 681.