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# Quantum Inverse Scattering Problem for Coupled Channels at Fixed Energy

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**Abstract.** In this short contribution we discuss certain aspects of the inverse scattering problem for coupled channels at fixed energy. The solution of the inverse problem means the calculation of the coupling potentials from the knowledge of the $S$-matrix obtained from nuclear reactions. We extend the modified Newton-Sabatier method known from the inverse problem of elastic scattering at fixed energy to the case of coupled channels.

## 1 Introduction

The calculation of potentials and coupling potentials from the phase shifts and from the $S$-matrix is denoted as the quantum inverse scattering problem. This procedure leads from the measured cross sections in nuclear physics or other fields in physics to the interaction potentials which are causally responsible for the results of scattering experiments. One distinguishes quantum inverse scattering problems at fixed angular momentum (the phase shifts or the $S$-matrix are functions of the incident energy) and at fixed energy (the phase shifts or $S$-matrix are functions of the quantum number of angular momentum). Usually only elastic potentials resulting from phase shifts are studied and various methods were developed for solving the elastic inverse scattering problem *e.g.*at fixed energy: the Newton-Sabatier method [1], the Bargmann method [2], the Ramm method [3], the finite difference method [4], the Cox-Thompson method [5, 6].

Inelastic scattering is a coupled channel problem. An example for nuclear inelastic scattering is the scattering of $^{12}$C on $^{12}$C with the excitation of the states of $^{12}$C. The starting point for the inversion of the coupled channel problem is the asymptotic wave function depending on the $S$-matrix. Here we want to review the solving of the inverse scattering problem for coupled channels at fixed energy within the modified Newton-Sabatier method for special cases of the coupling potentials. The general case of coupling potentials is yet waiting for a solution. The described methods are explained in detail in Ref. [7]. An approximate, but more practical method was found within the method of Ramm by using the first Born approximation [3, 8].

In Section 2 we explain the coupled channel equations, in Section 3 we introduce the modified Newton-Sabatier method and discuss transparent potentials, the special case that the interaction has a monopole character, and the interaction potential in the limes that the quantum number of the angular momentum approaches infinity. Finally, Section 4 gives the Conclusion.

## 2 Coupled Channel Equations

We introduce the relative coordinate $\mathbf{r}$ between the nuclei and a set of intrinsic coordinates $\xi$ for both nuclei. The Hamitonian of the system consists of the kinetic energy of the relative motion, $\hat{T}(\boldsymbol{r})$, of the intrinsic Hamiltonian, $\hat{h}(\xi)$, and the interaction energy, $\hat{W}(\boldsymbol{r},\xi)$,

$$\hat{H}(\boldsymbol{r},\xi) = \hat{T}(\boldsymbol{r}) + \hat{h}(\xi) + \hat{W}(\boldsymbol{r},\xi). \tag{1}$$

For the interaction energy we use a multipole expansion

$$\hat{W}(\boldsymbol{r},\xi) = \sum_{\lambda=0}^{\infty} \sum_{\mu=-\lambda}^{\lambda} Q_{\lambda\mu}(r,\xi) Y^*_{\lambda\mu}(\Omega) \tag{2}$$

with only integer values of $\lambda$ inculding the zero. The multipole moments $Q_{\lambda\mu}(r,\xi)$ describe the coupling between the relative and intrinsic motions. We do not consider nonlocal couplings and couplings to the spin degree of freedom of the nucleons. The intrinsic Hamiltonian $\hat{h}(\xi)$ has the eigenfunctions $\chi_{\nu JM}(\xi)$ and eigenvalues $\varepsilon_{\nu J}$:

$$\hat{h}(\xi)\chi_{\nu JM}(\xi) = \varepsilon_{\nu J}\chi_{\nu JM}(\xi), \tag{3}$$

where the quantum numbers $\nu$ characterize the intrinsic states, the quantum numbers $J$ and $M$ belong to the intrinsic angular momenta. Let us expand the eigenfunctions of (1) to the energy $E$ and quantum number $I$ of total angular momentum as follows:

$$\begin{aligned}\Psi^I_0(\boldsymbol{r},\xi) &= \sum_{\nu J} \sum_{\ell=|I-J|}^{I+J} \left[i^\ell Y_\ell(\Omega) \otimes \chi_{\nu J}(\xi)\right]^I_0 R^I_{\ell\nu J,n}(r)/r \\ &= \sum_{\nu J} \sum_{\ell m} i^\ell (\ell m J - m | I 0) Y_{\ell m}(\Omega) \chi_{\nu J-m}(\xi) R^I_{\ell\nu J,n}(r)/r. \end{aligned} \tag{4}$$

Then after projecting with the spherical harmonics and the intrinsic wave functions we obtain the coupled equations for the radial functions

$$\begin{aligned}\left(-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{\ell_\alpha(\ell_\alpha+1)\hbar^2}{2\mu r^2} + \varepsilon_\alpha - E\right) R^I_{\alpha n}(r) \\ = -\sum_\beta \sum_\lambda C^{\lambda I}_{\alpha\beta} v^\lambda_{\nu_\alpha J_\alpha,\nu_\beta J_\beta}(r) R^I_{\beta n}(r).\end{aligned} \tag{5}$$

Here, we introduced a set of quantum numbers $\alpha = \{\ell_\alpha, \nu_\alpha, J_\alpha, \varepsilon_\alpha\}$ to characterize the channels for fixed $I$. The coupling potentials are given by the product of the coefficients

$$C^{\lambda I}_{\alpha\beta} = i^{\ell_\beta - \ell_\alpha} \frac{(-1)^{J_\alpha + I + \lambda}}{\sqrt{4\pi}} \sqrt{(2\ell_\alpha + 1)(2\lambda + 1)(2\ell_\beta + 1)} \times \begin{pmatrix} \ell_\alpha & \lambda & \ell_\beta \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} \ell_\alpha & \lambda & \ell_\beta \\ J_\beta & I & J_\alpha \end{Bmatrix}, \quad (6)$$

with the reduced matrix elements

$$v^{\lambda}_{\nu_\alpha J_\alpha, \nu_\beta J_\beta}(r) = \langle \chi_{\nu_\alpha J_\alpha}(\xi) || Q_\lambda(r, \xi) || \chi_{\nu_\beta J_\beta}(\xi) \rangle. \quad (7)$$

For the solution of the inverse problem we need the condition that the coupling potentials vanish for large $r$-values, *i.e.* $r > a$, with exemption of a Coulomb-potential which can be quite easily taken into account (see Ref. [9]). The radial wave function can be written for $r > a$ with the $S$-matrix determined by the various differential cross sections:

$$R^I_{\alpha\beta}(r > a)/r = \sum_\gamma A_{\alpha\gamma} \frac{\sqrt{k_\gamma k_\beta}}{2} \Big( (S^I_{\gamma\beta} + \delta^I_{\gamma\beta}) j_{\ell_\gamma}(k_\gamma r) + i(S^I_{\gamma\beta} - \delta^I_{\gamma\beta}) n_{\ell_\gamma}(k_\gamma r) \Big), \quad (8)$$

where $k_\gamma = \sqrt{2\mu(E - \varepsilon_\gamma)/\hbar^2}$ and $\delta^I_{\gamma\beta} = \delta_{\ell_\gamma \ell_\beta} \delta_{J_\gamma J_\beta} \Delta(\ell_\gamma J_\gamma I) \Delta(\ell_\beta J_\beta I)$.

## 3 Newton-Sabatier Method for Multipole Coupling

We discuss the modified Newton-Sabatier method. First we write the coupled equations in a simpler form

$$\sum_\beta D^{VI}_{\alpha\beta}(r) R^I_{\beta n}(r) = \ell_\alpha(\ell_\alpha + 1) R^I_{\alpha n}(r) \quad (9)$$

with

$$D^{VI}_{\alpha\beta}(r) = r^2 \frac{2\mu}{\hbar^2} \left\{ \left[ \frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + E - \varepsilon_\alpha \right] \delta^I_{\alpha\beta} - V^I_{\alpha\beta}(r) \right\} \quad (10)$$

and

$$V^I_{\alpha\beta}(r) = \sum_\lambda C^{\lambda I}_{\alpha\beta} v^\lambda_{\nu_\alpha J_\alpha, \nu_\beta J_\beta}(r). \quad (11)$$

In a similar way we define a set of analogous coupled differential equations for the reference wave functions $R^{0I}_{\alpha n}(r)$ with an arbitrary chosen reference potential matrix $V^{0I}_{\alpha\beta}(r)$ $(= V^{0I}_{\beta\alpha}(r))$ :

$$\sum_\beta D^{V^0 I}_{\alpha\beta}(r) R^{0I}_{\beta n}(r) = \ell_\alpha(\ell_\alpha + 1) R^{0I}_{\alpha n}(r). \quad (12)$$

The Newton-Sabatier method introduces a kernel function which transforms from the wave functions $R^{0I}_{\alpha n}(r)$ to the wave functions $R^{I}_{\alpha n}(r)$. This transformation is named Povzner-Levitan representation (for details see Ref. [1]):

$$R^{I}_{\alpha n}(r) = R^{0I}_{\alpha n}(r) - \sum_{\beta} \int_0^r K^{V^I V^{0I} I}_{\alpha\beta}(r,r') R^{0I}_{\beta n}(r') \frac{dr'}{r'^2}, \tag{13}$$

where the elements $K^{V^I V^{0I} I}_{\alpha\beta}(r,r')$ of an integral kernel matrix appear. If we insert the transformation (13) into the coupled equations (12) we obtain a coupled system of equations:

$$\begin{aligned} 0 = &\sum_{\beta} r^2 \frac{2\mu}{\hbar^2} \left\{ V^{I}_{\alpha\beta}(r) - V^{0I}_{\alpha\beta}(r) + \frac{2}{r}\frac{\hbar^2}{2\mu}\frac{d}{dr}\left[ \frac{K^{V^I V^{0I} I}_{\alpha\beta}(r,r)}{r} \right] \right\} R^{0I}_{\beta n}(r) \\ &+ \sum_{\beta,\gamma} \int_0^r \left\{ D^{VI}_{\alpha\beta}(r) K^{V^I V^{0I} I}_{\beta\gamma}(r,r') - D^{V^0 I}_{\gamma\beta}(r') K^{V^I V^{0I} I}_{\alpha\beta}(r,r') \right\} R^{0I}_{\gamma n}(r') \frac{dr'}{r'^2} \\ &+ \sum_{\gamma} \int_0^r K^{V^I V^{0I} I}_{\alpha\gamma}(r,r') \left( \ell_\gamma \left( \ell_\gamma + 1 \right) - \ell_\alpha (\ell_\alpha + 1) \right) R^{0I}_{\gamma n}(r') \frac{dr'}{r'^2}. \end{aligned} \tag{14}$$

Here we assumed

$$K^{V^I V^{0I} I}_{\alpha\beta}(r=0,r') = K^{V^I V^{0I} I}_{\alpha\beta}(r,r'=0) = 0. \tag{15}$$

From equation (14) the Newton-Sabatier method results. For that we assume that the kernel matrix fulfills the equations

$$\begin{aligned} \sum_{\beta} \left( D^{VI}_{\alpha\beta}(r) - \ell_\alpha(\ell_\alpha+1)\delta^{I}_{\alpha\beta} \right) K^{V^I V^{0I} I}_{\beta\gamma}(r,r') \\ = \sum_{\beta} \left( D^{V^0 I}_{\gamma\beta}(r') - \ell_\gamma(\ell_\gamma+1)\delta^{I}_{\gamma\beta} \right) K^{V^I V^{0I} I}_{\alpha\beta}(r,r'). \end{aligned} \tag{16}$$

Then the coupling potential can be calculated as follows

$$V^{I}_{\alpha\beta}(r) = V^{0I}_{\alpha\beta}(r) - \frac{2}{r}\frac{\hbar^2}{2\mu}\frac{d}{dr}\left[ \frac{K^{V^I V^{0I} I}_{\alpha\beta}(r,r)}{r} \right]. \tag{17}$$

The inversion method means a search for a suitable expansion of the kernel matrix with some first unknown coefficients. The kernel matrix has to fulfill (16). Then the expansion coefficients are determined by using the wave functions in the asymptotic region depending on the $S$-matrix. After that the wave functions and the kernel matrix are continued into the interaction region $r < a$ and finally, the potential matrix is calculated with (17).

### 3.1 Transparent Potentials

A possible expansion of the kernel matrix with the coefficients $c^I_{nn'}$ is the following which fulfills (16)

$$K^{V^I V^{0I} I}_{\alpha\beta}(r,r') = \sum_{n,n'} c^I_{nn'} R^I_{\alpha n}(r) R^{0I}_{\beta n'}(r'). \tag{18}$$

This kernel is inserted in the Povzner-Levitan representation

$$R^I_{\alpha n}(r) = R^{0I}_{\alpha n}(r) - \sum_{n'n''} R^I_{\alpha n'}(r) c^I_{n'n''} L^I_{n''n}(r) \tag{19}$$

with

$$L^I_{n''n}(r) = \sum_{\beta} \int_0^r R^{0I}_{\beta n''}(r') R^{0I}_{\beta n}(r') \frac{dr'}{r'^2}. \tag{20}$$

The kernel matrix leads to transparent potentials. This means potentials which are different from zero but with a $S$-matrix $S = 1$. To recognize this fact we set $V^{0I}_{\alpha\beta}(r) = 0$ and $R^{0I}_{\alpha n}(r)/r = \sqrt{k_\alpha} j_{\ell_\alpha(I)}(k_\alpha r)\delta^I_{\alpha n}$. Then (20) can be easily calculated:

$$L^I_{n''n}(r) = \delta^I_{n''n} \int_0^{k_n r} j^2_{\ell_n(I)}(x)dx, \tag{21}$$

where the value of $\ell_n$ depends on I and the intrinsic state. The integral is asymptotically finite:

$$L^I_{n''n}(r \to \infty) = \delta^I_{n''n} \frac{1}{2\ell_n + 1} \frac{\pi}{2}. \tag{22}$$

With the matrix $L^I_{n''n}(r)$ we obtain a linear system of equations for the wave functions $R^I_{\alpha n}$:

$$\sum_{n'} R^I_{\alpha n'}(r) M_{n'n}(r)/r = \sqrt{k_\alpha} j_{\ell_\alpha(I)}(k_\alpha r)\delta^I_{\alpha n} \tag{23}$$

with

$$M_{n'n}(r) = \delta^I_{n'n} + c^I_{n'n} \int_0^{k_n r} j^2_{\ell_n(I)}(x)dx. \tag{24}$$

The solution of (23) is

$$R^I_{\alpha n}(r)/r = \sqrt{k_\alpha} j_{\ell_\alpha(I)}(k_\alpha r) M^{-1}_{\alpha n}(r). \tag{25}$$

For $r \to \infty$ the matrices $M$ and $M^{-1}$ do not depend on $r$. Then the $S$-matrix of the wave functions is $S = 1$ which means that the potential matrix derived with the kernel matrix (18) is transparent independent of the set of coefficients $c^I_{nn'}$.

### 3.2 Monopole Coupling Potentials

As a first application of the method we consider the case that the potential interaction does not depend on the direction of the relative coordinate $\boldsymbol{r}$, but only on the internuclear distance $r$ besides the dependence on the internal coordinates $\xi$. Then the interaction matrix can not transfer angular momentum from the relative motion to the intrinsic states. It consists only of a monopole term $\lambda = 0$ and is independent of $I$ and $\ell$:

$$\begin{aligned} V^I_{\alpha\beta}(r) =& V_{\alpha\beta}(r) \\ &= \frac{1}{\sqrt{4\pi}} \frac{1}{\sqrt{2J_\alpha+1}} \langle \chi_{\nu_\alpha J_\alpha} || Q_0(r,\xi) || \chi_{\nu_\beta J_\beta} \rangle . \end{aligned} \tag{26}$$

Only channels are coupled which have the same quantum numbers $I$, $\ell_\alpha$ and $J_\alpha$. The differential operators $D^{VI}_{\alpha\beta}(r)$ and $D^{V^0 I}_{\alpha\beta}(r)$ are independent of $I$.

$$D^{VI}_{\alpha\beta}(r) = D^{V}_{\alpha\beta}(r), \qquad D^{V^0 I}_{\alpha\beta}(r) = D^{V^0}_{\alpha\beta}(r). \tag{27}$$

In that case a kernel which fulfills equations (16) can be written with an additional sum over $I$

$$K^{VV^0}_{\alpha\beta}(r,r') = \sum_{n,n',I} c^I_{nn'} R^I_{\alpha n}(r) R^{0I}_{\beta n'}(r') \tag{28}$$

with the yet unknown coefficients $c^I_{nn'}$ which are determined from the asymptotic wave functions containing the $S$-matrix. This procedure has already been tested and found to work quite well (see Refs. [10, 11]).

### 3.3 Interaction Potential for Very Large Angular Momenta

For large values of $I$ we can show that the interaction matrix (11) is independent of $I$. Since the coefficients $C^{\lambda I}_{\alpha\beta}$ in $V^I_{\alpha\beta}$ contain the quantum number $I$, we consider the expression

$$C^{\lambda}_{\alpha\beta} = \lim_{I\to\infty} C^{\lambda I}_{\alpha\beta}. \tag{29}$$

We introduce the difference $s_\alpha$ between the quantum numbers $I$ and $\ell_\alpha$, defined as $s_\alpha = I - \ell_\alpha$. Then the following formulas for large values of $I$

$$\begin{aligned} \begin{pmatrix} I-s_\alpha & \lambda & I-s_\beta \\ 0 & 0 & 0 \end{pmatrix} \approx (-1)^{(2I+\lambda-s_\alpha-s_\beta)/2} \Big(\frac{1}{2}\Big)^\lambda \frac{1}{\sqrt{2I}} \\ \times \frac{\big((s_\alpha - s_\beta + \lambda)!(s_\beta - s_\alpha + \lambda)!\big)^{1/2}}{\Big(\frac{s_\alpha - s_\beta + \lambda}{2}\Big)! \Big(\frac{s_\beta - s_\alpha + \lambda}{2}\Big)!}, \end{aligned} \tag{30}$$

$$\begin{Bmatrix} I-s_\alpha & \lambda & I-s_\beta \\ J_\beta & I & J_\alpha \end{Bmatrix} \approx \frac{(-1)^{J_\alpha+J_\beta+\lambda}}{\sqrt{2I}} \begin{pmatrix} J_\beta & \lambda & J_\alpha \\ s_\beta & s_\alpha - s_\beta & -s_\alpha \end{pmatrix} \tag{31}$$

lead to the coefficients

$$C^{\lambda}_{\alpha\beta} = i^{s_\alpha - s_\beta}(-1)^{(2J_\beta+\lambda-s_\alpha-s_\beta)/2}\left(\frac{1}{2}\right)^{\lambda}\frac{\left((s_\alpha-s_\beta+\lambda)!(s_\beta-s_\alpha+\lambda)!\right)^{1/2}}{\left(\frac{s_\alpha-s_\beta+\lambda}{2}\right)!\left(\frac{s_\beta-s_\alpha+\lambda}{2}\right)!} \times \frac{1}{\sqrt{4\pi}}\sqrt{2\lambda+1}\begin{pmatrix} J_\beta & \lambda & J_\alpha \\ s_\beta & s_\alpha - s_\beta & -s_\alpha \end{pmatrix}. \quad (32)$$

The inverse problem with angular momentum-independent interactions with coefficients $C^{\lambda}_{\alpha\beta}$ can be solved similarly to the method described in Section 3.2. In that case one can use the expansion (28) for the kernel matrix including the summation over $I$ if one replaces $\ell_\alpha(\ell_\alpha+1)$ by $I(I+1)$ in (9) and (12) (a similar approximation is used for the coefficients $C^{\lambda}_{\alpha\beta}$). Then the kernel matrix expansion (28) again fulfills the necessary equations (16).

In principle, one now has the possibility to solve the inverse scattering problem in the case of a general Hamiltonian (1) at fixed energy. The procedure is the following: Ramm proved that the potential in elastic scattering is determined already by an infinite subset of phase shifts at a fixed energy [12]. Let us assume that the same statement is also true for coupled channels and that an infinite subset of the scattering matrix already can determine the interaction matrix. Hence, we can choose an infinite subset of the $S$-matrix with quantum numbers $I$ of total angular momenta in the interval $I_{min} \leq I \leq I_{max} \to \infty$. If the condition that the absolute value of the difference $|C^{\lambda I_{min}}_{\alpha\beta} - C^{\lambda}_{\alpha\beta}|$ is smaller than an arbitrary small $\epsilon$ is fulfilled, we can solve the inverse problem with the subset of the $S$-matrix with $I \geq I_{min}$ and with the coefficients $C^{\lambda}_{\alpha\beta}$ using the expansion of the kernel matrix and the method described in Section 3.2. However, this method is not practicable since the $S$-matrix elements can not be extracted for high angular momenta from the experimental differential cross section, especially in reactions of nuclear physics.

## 4 Conclusion

A more practical, but approximate procedure to solve the inverse scattering problem inside the Newton-Sabatier method starts with the ansatz for the kernel matrix (see Ref. [9])

$$K^{V^I V^{0I} I}_{\alpha\beta}(r,r') = \sum_{\lambda} C^{\lambda I}_{\alpha\beta} k^{\lambda}_{\nu_\alpha J_\alpha, \nu_\beta J_\beta}(r,r') \quad (33)$$

with the coupling potentials obtained as

$$v^{\lambda}_{\nu_\alpha J_\alpha, \nu_\beta J_\beta}(r) = v^{0\lambda}_{\nu_\alpha J_\alpha, \nu_\beta J_\beta}(r) - \frac{2}{r}\frac{\hbar^2}{2\mu}\frac{d}{dr}\left[\frac{k^{\lambda}_{\nu_\alpha J_\alpha, \nu_\beta J_\beta}(r,r)}{r}\right]. \quad (34)$$

The kernels $k^{\lambda}_{\nu_\alpha J_\alpha,\nu_\beta J_\beta}(r,r')$ should not depend on $I$ and can be expanded in a sum over $I$ and the wave functions

$$k^{\lambda}_{\nu_\alpha J_\alpha,\nu_\beta J_\beta}(r,r') = \sum_{n,n',I} c^{\lambda I}_{nn'} R^{I}_{\alpha n}(r) R^{0I}_{\beta n'}(r'). \tag{35}$$

Then, the method of Section 3.2 can again be applied to calculate the interaction matrix. However, the essential disadvantage of this procedure is that the kernel (33) does not fulfill the necessary equations (16) also if the coefficients $C^{\lambda I}_{\alpha\beta}$ are independent of $I$, and the centrifugal term in (5) is approximated by $I(I+1)\hbar^2/(2\mu r^2)$.

Another approximate procedure is based on the first Born approximation (see Ref. [8]) and needs at least only one line of the $S$-matrix as initial data. It works under the conditions ($|V|$ is a mean value of the interaction):

$$\frac{\mu|V|a^2}{\hbar^2}\sqrt{ka} \ll 1 \quad \text{if} \quad ka < 1, \tag{36}$$

$$\frac{\mu|V|a}{\hbar^2 k} \ll 1 \quad \text{if} \quad ka > 1. \tag{37}$$

The starting point are integral equations for the wave functions derived from the coupled differential equations ($U^I_{\alpha\beta} = (2\mu/\hbar^2)V^I_{\alpha\beta}$)

$$R^I_{\alpha n}(r)/r = R^{0I}_{\alpha n}(r)/r - \imath \int_0^a r'^2 dr' k_\alpha j_{\ell_\alpha}(k_\alpha r_<) \times (j_{\ell_\alpha}(k_\alpha r_>) + \imath n_{\ell_\alpha}(k_\alpha r_>)) \sum_\beta U^I_{\alpha\beta}(r') R^I_{\beta n}(r')/r'. \tag{38}$$

If $R^I_{\beta n}(r')$ is replaced by $R^{0I}_{\beta n}(r')$ in first Born approximation, a relation between the moments of the potential matrix and $S$-matrix can be derived [8]

$$S^I_{\alpha\beta} - \delta^I_{\alpha\beta} = -2\imath\sqrt{k_\alpha k_\beta}\int_0^a r'^2 dr' j_{\ell_\alpha}(k_\alpha r') U^I_{\alpha\beta}(r') j_{\ell_\beta}(k_\beta r'). \tag{39}$$

From the moments of the $S$-matrix one gets the potential matrix by the method of Bachus and Gilbert [8]. In the case that the first Born approximation works (see equations (36) and (37)) the test calculations show good results [8]. In first Born approximation the matrix elements $S^I_{\alpha\beta}$ with fixed values of $\alpha$ and $\beta$ already give the potential matrix elements $v^{\lambda}_{\nu_\alpha J_\alpha \nu_\beta J_\beta}(r)$. This is a great advantage for the inversion since most experiments provide differential cross sections which only yield parts of the complete $S$-matrix.

Up to know the presented methods were not yet applied to realistic cases. We plan to use the inversion procedures for examples of heavy ion scattering, *e.g.*for the study of the effects of nuclear molecular states in the scattering of $^{12}$C on $^{12}$C. Such a work has as necessary condition that trustworthy $S$-matrices can be extracted from the experimental data. But that is another severe problem.

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