

# Symmetries of Collective Models in Intrinsic Frame

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**Abstract.** In the paper a very general definition of intrinsic frame, by means of group theoretical methods, is introduced. It allows to analyze nuclear properties which are invariant in respect to the group which defines the intrinsic frame. For example, nuclear shape is a well determined feature in the intrinsic frame defined by the Euclidean group. It is shown that using of intrinsic frame gives an opportunity to consider intrinsic nuclear symmetries which are independent of symmetries observed in the laboratory frame. An importance of the notion of partial symmetries is emphasized.

## 1 Introduction

A problem of possible existence of nuclear symmetries corresponding to high order groups considered in nuclear intrinsic frame of reference has been predicted in earlier publications [1]. There is considered an example of tetrahedral symmetries which is expected to produce large shell gaps in the single particle spectra of some nuclei. The characteristic for this point groups – four-fold degeneracy – increases the average level spacing [2, 3]. This leads to specific tetrahedral-magic shell-closures for nucleon numbers 32, 40, 56, 64, 70, 90-94, 112, and 136-138. There were a few experiments related to the problem of “tetrahedral”, *e.g.*, Ref. [4, 5]. Similarly, in the Rare Earth nuclei such as <sup>152,156</sup>Gd, <sup>154,156</sup>Dy, <sup>164</sup>Er, <sup>164</sup>Yb, but also in the Actinides in <sup>230–234</sup>U, there are some signature which support the idea of “tetrahedral” nuclei, however, the results are not unique [6, 7] and requires further experiments.

The intrinsic geometrical symmetries are related to an idea of intrinsic frame generated by the Euclidean E(3) group. This kind of intrinsic coordinates allows to define such properties like position of a nucleus in the space (the center of mass), nuclear shape (deformation) and the orientation of a nucleus in respect to the laboratory frame and similar features related to its geometrical properties.

There is a series of papers devoted to different definitions and properties of an intrinsic frame. In nuclear physics the most interesting frame is the frame which is attached to a nucleus. A rather general definition of the fixed-body frame is described, *e.g.*, by Biedenharn and Louck [9].

Another idea of intrinsic frame is based on a separation of various types of quantum motions. It was successfully analyzed by Eckart in molecular physics [8]. It can be shown, however, that the Eckart condition is restricted only to potentials with a single minimum. A generalization of his condition, which can be used in case of potential energy with more than one minimum, was formulated by Sayvetz [10, 11].

However, it was proved by Guichardet [12] that both kinds of motion, i.e. rotation and vibration, cannot be, in general case, separated exactly. Another interesting approach to an intrinsic frame and internal quantum motions was done by Iwai [13], however, the method described in his paper can be applied, in practice, only to a few-body systems. In this paper we follow a group theoretical definition of the intrinsic frame, see [14].

## 2 Intrinsic Frame

A general definition of an intrinsic frame where a part of intrinsic coordinates is defined as the group parameters can be summarized as follows:

- Let  $q^{lab} = (q_1^{lab}, q_2^{lab}, \dots, q_s^{lab})$  denotes the laboratory variables of the nuclear model and  $q = (q_1, q_2, \dots, q_s)$  their counterparts which will be defined below.
- Let  $G$  denotes a Lie group acting by the operators  $\hat{g}$  in the space of nuclear degrees of freedom  $q^{lab}$ . The  $r$  parameter group  $G \ni g$  is parameterized by  $r$  parameters  $g(\theta) = g(\theta_1, \theta_2, \dots, \theta_r)$ . The operator  $\hat{g}$  represents the element  $g \in G$ . The group  $G$  is chosen to describe (approximately) some important nuclear motions. The group parameters  $\theta$  are intended to be used as a part of intrinsic variables.
- One defines the transformation formula from the laboratory to an  $G$ -intrinsic frame (and partially intermediate intrinsic variables  $q$ ) by the group transformations  $\hat{g}: q^{lab} \rightarrow q$ :

$$q = \hat{g} q^{lab}. \quad (1)$$

The action (1) fulfils the group composition requirement

$$\hat{h} q = \hat{h}(\hat{g} q^{lab}) = (\widehat{hg}) q^{lab} = \widehat{(hg)} q^{lab} \quad (2)$$

for every  $g, h \in G$ . By definition (1) the intrinsic variables  $q$  are invariant in respect simultaneous action  $\hat{h}\hat{h}_G$ , where  $h \in G$ . The first operator  $\hat{h}$  acts on nuclear degrees of freedom  $q^{lab}$  according to the formula (2) and the second operator  $\hat{h}_G$  acts only on the group manifold of the group  $G$  as the left shift operation

$$\hat{h}\hat{h}_G q = \hat{h}(\widehat{(h^{-1}g)} q^{lab}) = \hat{h}\hat{h}^{-1}\hat{g} q^{lab} = \hat{g} q^{lab} = q. \quad (3)$$

This operation corresponds to simultaneous shift of the intrinsic frame and the laboratory variables by the same transformation which results in unchanged variable  $q$ . Other words, this property makes the intrinsic variables rigid in respect to the motions (transformations) generated by the chosen group  $G$ .

- At this stage we have too many intrinsic variables:  $s$  variable  $q$  and  $r$  variables  $\theta$  parameterizing the group  $G$ . Let us assume, in the laboratory frame we have  $s > r$  degrees of freedom (the case  $s \leq r$  should be considered separately). It requires  $r$  additional conditions which exclude  $r$  intrinsic variables  $q$  giving a set of independent variables containing  $s - r$  intrinsic variables of the  $q$ -type and  $r$  group parameters (now intrinsic variables)  $\theta$ . A choice of the additional conditions:

$$F_i(q, \theta) = 0, \quad (4)$$

where  $i = 1, 2, \dots, r$ , is arbitrary but in practice it defines often physical properties of remaining variables  $q$ .

A standard example of introducing an intrinsic frame in nuclear physics is a rotating frame fixed to a nucleus. A systematic description of the nuclear surface can be done by multipole surface deformation variables  $\alpha_{\lambda\mu}^{lab}$ , where in general  $\lambda = 0, 1, 2, \dots$  and  $\mu = -\lambda, -\lambda + 1, \dots, \lambda$ . They are spherical tensors which transform in respect to the rotation group  $g \in SO(3)$  according to the standard rule:

$$\hat{g} \alpha_{\lambda\mu}^{lab} = \sum_{\mu'} D_{\mu'\mu}^{\lambda}(g) \alpha_{\lambda\mu'}^{lab}, \quad (5)$$

where the group elements  $g = g(\Omega)$  are parameterized by Euler angles  $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ . The functions  $D_{\mu'\mu}^{\lambda}(\Omega)$  denote Wigner functions for the rotation group [16]. Now we can define the 'rotating', intrinsic, counterparts of the laboratory deformation variables  $\alpha_{\lambda\mu}^{lab}$ :

$$\alpha_{\lambda\mu} = \sum_{\mu'} D_{\mu'\mu}^{\lambda}(\Omega) \alpha_{\lambda\mu'}^{lab}. \quad (6)$$

According to our consideration there are needed 3 additional conditions (the group  $SO(3)$  has 3 parameters) which in many cases are chosen following the Bohr suggestion for quadrupole case:

$$F_1(\alpha, \Omega) = \alpha_{21} = 0, \quad F_2(\alpha, \Omega) = \alpha_{2,-1} = 0, \quad F_3(\alpha, \Omega) = \alpha_{22} - \alpha_{2,-2} = 0. \quad (7)$$

These definition introduces the following set of of intrinsic collective variables in the rotating frame:  $\{\alpha_{1\mu}, \alpha_{20}, \alpha_{22}, \alpha_{3\mu}, \dots, \Omega\}$ . In pure quadrupole case the conditions (7) define, often used, a principal axes frame in respect to a quadrupole shape.

In quantum case the rotating frame has a bit different meaning as in the classical mechanics. The quantum rotation is not longer a dependence of the orientation of a nucleus (body) in respect to the laboratory fixed frame as a function of time. It is a quantum motion in which one can only give a probability of finding the potential space orientation of a nucleus being in a given quantum state. In the figure (1) one can find the probability distribution of the spin (nucleus) orientation. The points on the figure are drawn by the vector having direction of the spin and length equal to a probability of finding this orientation. These

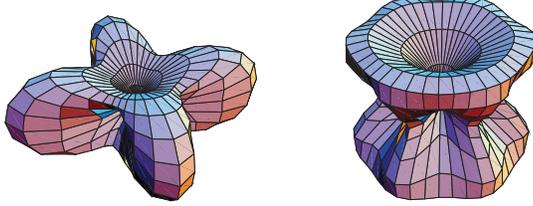


Figure 1. The spin orientation probability for a rotating system. The wave functions:  $\psi \sim D_{M2}^5(\Omega) - D_{M,-2}^5(\Omega)$  (left) and  $\psi \sim D_{M3}^5(\Omega) - D_{M,-3}^5(\Omega)$  (right)

examples shows that the quantum rotating nucleus (body) is moving randomly according to the angular part of a quantum state describing the nucleus.

In the remaining part of the paper we consider only a class of rotating frames. A general case can be analyzed in a similar way.

The details of the construction of an intrinsic frame are more intuitive considering single particle variables of the  $A$ -nucleon system. Let us denote by

$$\vec{x}_n^{lab} = (x_{n1}^{lab}, x_{n2}^{lab}, x_{n3}^{lab}), \quad (8)$$

where  $n = 1, \dots, A$ , the radius raw-vector of  $n$ -th nucleon in the laboratory frame. The position vector in the configuration space for  $A$  nucleons can denoted as:

$$x = (\vec{x}_1, \vec{x}_2, \vec{x}_3, \dots, \vec{x}_A). \quad (9)$$

and the laboratory frame basic, orthonormal unit vectors by:

$$\vec{l}_1, \vec{l}_2, \vec{l}_3. \quad (10)$$

To have a physical relation between the laboratory and the intrinsic (rotating) frame the action (1) of the Euclidean group  $E(3) \ni \hat{g}(\vec{a}, \Omega)$  can be defined as:

$$\vec{x}'_n = \hat{g}(\vec{a}, \Omega) \vec{x}_n = (\vec{x}_n - \vec{a}) R(\Omega^{-1}), \quad (11)$$

where  $n=1,2,\dots,A$ , 3-translation vector is  $\vec{a} = (a_1, a_2, a_3)$  and the Euler angles describing orientation of a nucleus in the laboratory frame are  $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ .

$R(\Omega)$  denotes the rotation matrix in  $\mathbb{R}^3$ . This action determines the following transformation from the laboratory to the intrinsic frame  $x^{lab} \rightarrow x$ :

$$\vec{x}_n = \hat{g}(\vec{a}, \Omega) \vec{x}_n^{lab} = (\vec{x}_n^{lab} - \vec{a})R(\Omega^{-1}). \quad (12)$$

The Euclidean group has 6 parameters and require 6 additional conditions. The first 3 conditions,  $k = 1, 2, 3$ :

$$F_k(x, \vec{a}, \Omega) = \frac{1}{\sum_{n=1}^A m_n} \sum_{n=1}^A m_n x_{nk} = 0 \quad (13)$$

identify the translation vector  $\vec{a}$  with the center of mass system  $\vec{R}_{cm}$ . To show this it is enough to sum up the equations (12) and use the conditions (13):

$$\begin{aligned} \vec{0} &= \frac{1}{\sum_{n=1}^A m_n} \sum_{n=1}^A m_n \vec{x}_n \\ &= \frac{1}{\sum_{n=1}^A m_n} \sum_{n=1}^A m_n (\vec{x}_n^{lab} - \vec{a})R(\Omega^{-1}) = (\vec{R}_{cm} - \vec{a})R(\Omega^{-1}). \end{aligned} \quad (14)$$

Multiplying both sides from the right by the rotation matrix  $R(\Omega)$  one gets  $\vec{a} = \vec{R}_{cm}$ . It means that we introduce 3 collective variables  $\vec{a}$  which in the intrinsic frame describe translational motion of the nucleus.

To simplify notation we assume that the intrinsic frame is the center of mass frame i.e.  $\vec{a} = 0$ .

The next 3 conditions are traditionally related to the traceless quadrupole mass tensor:

$$Q_{ij}^{lab} \equiv \sum_{n=1}^A m_n (3x_{ni}^{lab} x_{nj}^{lab} - \delta_{ij} \|\vec{x}_n^{lab}\|^2) \quad (15)$$

Diagonalization of  $Q_{ij}^{lab}$  by an orthogonal matrix  $D$  allows to define principal axes of a body

$$D(\Omega^{-1})^T Q^{lab} D(\Omega^{-1}) = \text{diag}(Q_1, Q_2, Q_3) \quad (16)$$

in which the mass quadrupole tensor and, at the same time, the moment of inertia are diagonal. For  $Q_1 \neq Q_2 \neq Q_3 \neq Q_1$  the condition (16) defines 3 Euler angles uniquely. In terms of spherical components of the mass quadrupole tensor the condition (16) can be rewritten as:

$$\begin{aligned} F_k(x, \vec{a}, \Omega) &= Q_{\pm 1}^{(2)}(x) = 0 \text{ for } k = 4, 5 \\ F_6(x, \vec{a}, \Omega) &= Q_2^{(2)}(x) - Q_{-2}^{(2)}(x) = 0. \end{aligned} \quad (17)$$

This conditions relate the orientation of a nucleus with 3 Euler angles parameterizing the Euclidean group  $E(3)$ . They define three collective variables which describe nuclear rotational motion.

### 3 Symmetrization Group

The transformations from the laboratory to the intrinsic frame are, in general, not unique 2. In many cases, to keep physical interpretation of intrinsic variables we are not able to restrict their domains to make these transformations one-to-one functions:

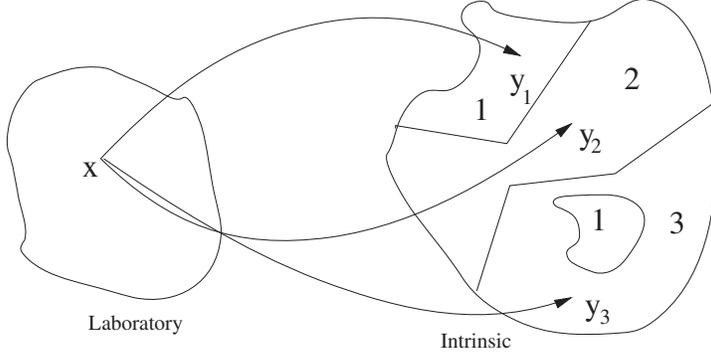


Figure 2. A one-to-many transformation from laboratory to intrinsic variables.

A solution of the problem is to use a wider space of nuclear variables, with repeated domains of uniqueness of the transformations from the laboratory to the intrinsic frame, however, as a consequence, the space of physical states has to fulfil an additional condition. It leads to a concept of symmetrization group [15].

In fact, one can find a group of transformations  $h \in \overline{G}_s$  acting on the intrinsic variables (an intrinsic group, see definition in the section (4)):

$$(x, \vec{a}, \Omega) \xrightarrow{h} (x', \vec{a}', \Omega') \quad (18)$$

which leave invariant the corresponding laboratory coordinates:

$$x^{lab}(x', \vec{a}', \Omega') = x^{lab}(x, \vec{a}, \Omega). \quad (19)$$

It implies that for a given state  $\Psi(x^{lab})$

$$\Psi(x^{lab}) = \Psi(x, \vec{a}, \Omega) = \Psi(x', \vec{a}', \Omega'), \quad (20)$$

i.e. the state defined in the intrinsic frame has to be invariant in respect to all transformations belonging to the group  $\overline{G}_s$ . The group  $\overline{G}_s$  is called the symmetrization group. The symmetrization condition requires that for all  $h \in \overline{G}_s$ :

$$\overline{T}(h)\Psi(x, \vec{a}, \Omega) = \Psi(x, \vec{a}, \Omega), \quad (21)$$

where for  $h \in \overline{G}_s$  the operators  $\overline{T}(h)$  represent action of the intrinsic group  $\overline{G}_s$  in the space of states expressed in terms of intrinsic variables, for detailed definition see (37).

The symmetrization group for the considered example of rotating intrinsic frame is the octahedral point group acting on the intrinsic variables. In general, for every particular choice of intrinsic variables the symmetrization group has to be constructed separately. For example, for a non-standard choice of quadrupole collective intrinsic variables  $(\alpha_{20}, \alpha_{21}, \Omega)$ . The additional 3 conditions are written as  $F_{1,2}(\alpha, \Omega) = \alpha_{2\pm 2} = 0$  and  $F_3(\alpha, \Omega) = \alpha_{21} + \alpha_{2-1} = 0$ . It gives the following equations for elements—rotations of the symmetrization group determined by the Euler angles  $\omega = (\omega_1, \omega_2, \omega_3)$  and the space inversion operation:

$$\begin{aligned} D_{\pm 20}^2(\omega) &= 0 \\ D_{\pm 2,1}^2(\omega) - D_{\pm 2,-1}^2(\omega) &= 0 \\ D_{10}^2(\omega) + D_{-1,0}^2(\omega) &= 0 \\ D_{11}^2(\omega) - D_{1,-1}^2(\omega) &= D_{-1-1}^2(\omega) - D_{-11}^2(\omega). \end{aligned} \quad (22)$$

The resulting group is a dihedral point group  $\overline{D}_{2h}$ . As we see the symmetrization groups can be very different.

The requirement of symmetrization has to be fulfilled by the intrinsic Hamiltonian itself:

$$\overline{T}(h) \mathcal{H} \overline{T}(h)^{-1} = \mathcal{H}, \quad (23)$$

for all  $h \in \overline{G}_s$ . In addition, the action of the intrinsic Hamiltonian should be closed within the subspace of symmetrized functions:

$$\mathcal{K} = \{\psi: \overline{T}(h)\psi = \psi \text{ for all } h \in \overline{G}_s\}. \quad (24)$$

This condition can be formulated as follows:

$$\mathcal{H}\psi \in \mathcal{K} \quad \text{for all } \psi \in \mathcal{K}, \quad (25)$$

because only in this case we are working in the physical subspace in which every intrinsic state has a unique counterpart in the laboratory frame.

In practice, one encounters a few approaches to the problem of symmetrization of intrinsic nuclear Hamiltonian, see also [15]:

### **0. Already Symmetrized.**

The Hamiltonian  $\hat{\mathcal{H}}$  is constructed directly to be invariant in respect to an appropriate symmetrization group. The action of  $\hat{\mathcal{H}}$  is closed within the physical subspace  $\mathcal{K}$ .

### **1. Projection.**

The appropriate Hamiltonian  $\hat{\mathcal{H}}_1$  is obtained from a non-invariant initial Hamiltonian  $\hat{\mathcal{H}}$  by projection onto the space of symmetrized functions  $\mathcal{K}$ :

$$\hat{\mathcal{H}}_1 = P_{\mathcal{K}} \hat{\mathcal{H}} P_{\mathcal{K}}, \quad (26)$$

where  $P_{\mathcal{K}}$  is the projection operator onto the space  $\mathcal{K}$ . In general it can be written (for discrete, finite symmetrization group  $\overline{G}_s$ ):

$$P_{\mathcal{K}} = \frac{1}{\text{card}(\overline{G}_s)} \sum_{g \in \overline{G}_s} g, \quad (27)$$

where  $\text{card}(\overline{G}_s)$  denotes the order (number of elements) of the symmetrization group.

The constructed Hamiltonian  $\hat{\mathcal{H}}_1$  fulfills required properties of physical Hamiltonian. It can be rewritten by means of the spectral theorem:

$$\hat{\mathcal{H}}_1 = \sum_{\nu} E_{1;\nu} |\Psi_{1;\nu}\rangle \langle \Psi_{1;\nu}|. \quad (28)$$

Hamiltonian  $\hat{\mathcal{H}}_1$  has the symmetry  $\overline{G}_s$  or larger, independently of the symmetry of the original Hamiltonian  $\hat{\mathcal{H}}$ . The eigenvalues and eigenvectors of  $\hat{\mathcal{H}}_1$ , in general, differs from the initial Hamiltonian  $\hat{\mathcal{H}}$ .

## 2. Selection.

First one needs to solve the eigenequation for the  $\overline{G}_s$  non-invariant Hamiltonian  $\hat{\mathcal{H}}$  in the full space of intrinsic functions  $\mathcal{K}_{full}$  (the space of all square integrable functions of intrinsic variables):

$$\hat{\mathcal{H}}|\Psi_n\rangle = E_n|\Psi_n\rangle. \quad (29)$$

Next, one needs to choose solutions which fulfill the symmetrization condition:

$$P_{\mathcal{K}}|\Psi_n\rangle = |\Psi_n\rangle \equiv |\Psi_n\rangle_{\mathcal{K}}. \quad (30)$$

The spectral decomposition of the resulting intrinsic Hamiltonian is:

$$\hat{\mathcal{H}}_2 = \sum_n E_n |\Psi_n\rangle_{\mathcal{K}} \langle \Psi_n|, \quad (31)$$

where summation is only over selected symmetrized eigenstates of the initial Hamiltonian.

One can show, the eigensolutions of the Hamiltonian  $\hat{\mathcal{H}}_2$  are also eigensolutions of the first Hamiltonian  $\hat{\mathcal{H}}_1$ . The opposite property is, generally, not fulfilled [15].

One needs to notice a typical example of collective nuclear models: the original Bohr Hamiltonian (five dimensional harmonic oscillator) and its different modifications are not invariant in respect to the symmetrization group (the appropriate octahedral point group,  $\overline{G}_s = \overline{O}$ ). In this context they are unphysical Hamiltonians.

Then, to recover symmetrization, the second procedure is used, i.e. selection. The Bohr's type of the collective models use the eigenstates which are invariant in respect to  $\overline{G}_s$  and which are chosen from the full set of original Hamiltonian.

## 4 Intrinsic Groups

It has been shown, that structure of intrinsic space is more complicated and it is constrained by the artificial symmetry group  $\overline{G}_s$ . The same one can say about

any intrinsic Hamiltonian which has to be invariant in respect to the symmetrization group, which, in fact, is not its physical symmetry. On the other hand the symmetries defined in the intrinsic frame are independent of these which are defined in the outer, laboratory frame. The best tool for analysis of symmetries in an intrinsic frame, it is to define, besides of the transformation groups acting in laboratory frame, the intrinsic groups acting in the corresponding intrinsic frame. The definition of intrinsic groups is based on the so called left and right shift operations on the group manifolds, however, for physical applications the definition given in the textbook [17] is more appropriate.

Def. For each element  $g$  of the group  $G$ , one can define a corresponding operator  $\bar{g}$  in the group linear space  $\mathcal{L}_G$  as:

$$\bar{g}S = Sg, \quad \text{for all } S \in \mathcal{L}_G. \quad (32)$$

The group formed by the collection of the operators  $\bar{g}$  is called the intrinsic group  $\bar{G}$  of  $G$ .

The most important properties are:

- Both, the laboratory and its intrinsic counterpart commute:

$$[G, \bar{G}] = 0. \quad (33)$$

It means that laboratory and intrinsic symmetries are independent, as it was already mentioned.

- The groups  $G$  and  $\bar{G}$  are anti-isomorphic. It allows to use all properties of the laboratory group like, e.g. representations, Clebsch-Gordan coupling coefficients and others, to intrinsic groups.

As an example, one can again consider the most popular type of intrinsic frame, the rotating frame and the corresponding intrinsic group  $\bar{G} = \overline{\text{SO}(3)}$ . Let us come back to an example of the  $A$ -nucleon system and denote by  $\hat{h}(\Omega)$ , where  $h = h(\Omega) \in \text{SO}(3)$ , the rotation operator in the laboratory frame. Here, the symbol  $\Omega$  denotes, as before, Euler angles of a rotation  $h$ . The action of the rotation intrinsic group  $g \in \overline{\text{SO}(3)}$  onto intrinsic nucleon coordinates  $\{x_{nk}\}$  can be written as:

$$x_{nk} = \hat{h}(\Omega)x_{nk}^{lab} = \sum_{k'} R_{k'k}(\Omega)x_{nk'}^{lab} \quad (34)$$

$$x'_{nk} = \hat{g}x_{nk} = \sum_{k'} R_{k'k}(g^{-1})x_{nk'} \quad (35)$$

$$\Omega' = \hat{g}\Omega = \Omega g, \quad (36)$$

where  $R(\Omega)$  represents the rotation matrix in three dimensional Euclidean space an  $\Omega g$  is composition of two rotations  $\Omega$  and  $g$ .

They define an action of the intrinsic group  $\overline{\text{SO}(3)} \ni g$  in the space of functions of intrinsic variables:

$$\bar{T}(g)\psi(x, \Omega) = \psi(\hat{g}x, \Omega g^{-1}). \quad (37)$$

However, to have action which does not jump out of the set of intrinsic variables the additional conditions (4) have to be taken into account. Instead of general consideration we can show the problem on the example of the rotating body-fixed frame (17). One needs to find the maximal subgroup of  $\overline{\text{SO}(3)}$  which does not change the principal axes condition:  $Q_{\pm 1}^{(2)}(x) = 0$ ,  $Q_2^{(2)}(x) = Q_{-2}^{(2)}(x)$ .

Because the quadrupole tensor transforms under the intrinsic rotation group as:

$$Q_{\mu}^{\prime(2)}(x) = \hat{g}Q_{\mu}^{(2)}(x) = \quad (38)$$

$$D_{0\mu}^2(g^{-1})Q_0^{(2)}(x) + [D_{-2\mu}^2(g^{-1}) + D_{2\mu}^2(g^{-1})] Q_2^{(2)}(x) \quad (39)$$

The requirements  $Q_{\pm 1}^{\prime(2)}(x) = 0$  and  $Q_2^{\prime(2)}(x) = Q_{-2}^{(2)}(x)$  lead to the following equations for allowed intrinsic rotations in the intrinsic variables

$$\begin{aligned} D_{0,\pm 1}^2(g^{-1}) &= 0 \\ D_{-2\pm 1}^2(g^{-1}) + D_{2\pm 1}^2(g^{-1}) &= 0 \\ D_{02}^2(g^{-1}) - D_{0,-2}^2(g^{-1}) &= 0 \\ D_{-2,-2}^2(g^{-1}) + D_{2,-2}^2(g^{-1}) &= D_{-22}^2(g^{-1}) + D_{22}^2(g^{-1}). \end{aligned} \quad (40)$$

Solutions of the equations (40) can be identified with all intrinsic rotations which collection results in an octahedral point group acting in the intrinsic variables  $(x, \Omega)$ . This statement is true under assumption that both  $Q_0^{(2)}(x)$  and  $Q_2^{(2)}(x)$  are independent.

It gives an important hint: **due to the additional conditions required to define an intrinsic frame not all rotations are allowed in the intrinsic frame.**

The same remark is true for a general case.

There arise a question which operations are allowed in the intrinsic variables space ? A general answer is that all operations which leave the conditions (4) invariant do not violate the structure of the intrinsic space.

On the other hand, if we construct the set of independent (not all variables  $q$ ) intrinsic variables which consists of  $\{\xi, \theta\}$ , where  $\xi = \xi(q)$ , then any one-to-one transformation  $(\xi, \theta) \rightarrow (\xi', \theta')$  is an allowed transformation which can be potentially a symmetry operation. It is a pity, however, further analysis of allowed transformation depends on a model under considerations.

## 5 Partial Symmetries – Non-Orthogonal Decomposition

To simplify notation, let us consider the quadrupole+octupole model of intrinsic Hamiltonian without coupling between collective vibrations and rotations,

where the intrinsic collective variables  $\alpha_{20}, \alpha_{22}, \{\alpha_{3\mu}\}, \Omega$  are determined by the shape of nuclear surface :

$$R(\vartheta, \phi) = R_0 \left( 1 + \alpha_{20} Y_{20}(\vartheta, \phi) + \alpha_{22} (Y_{22}(\vartheta, \phi) + Y_{2,-2}(\vartheta, \phi)) + \sum_{\mu=-3}^3 \alpha_{3\mu}^* Y_{3\mu}(\vartheta, \phi) \right). \quad (41)$$

Extension to general case is straightforward. Lack of coupling between commuting vibrational and rotational terms implies the following structure of the intrinsic Hamiltonian

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_{vib} + \hat{\mathcal{H}}_{rot} = T_{vib;2} + T_{vib;3} + V_{vib} + \hat{\mathcal{H}}_{rot}, \quad (42)$$

where  $T$  denote vibrational quadrupole and octupole kinetic energy,  $V$  is responsible for a potential energy and  $\hat{\mathcal{H}}_{rot}$  describes a generalized rotor Hamiltonian (the operator built as a sum of polynomials constructed from a set of intrinsic angular momentum operators) [18, 19].

Because the Hamiltonian is expressed in intrinsic variables it is invariant in respect to laboratory rotations, as should be. It implies the angular momentum quantum numbers  $J, M$  to be good quantum numbers. The formal intrinsic symmetry  $\overline{G}_H$  of the Hamiltonian  $\hat{\mathcal{H}}$  is a direct product of symmetry groups for vibrational and rotational terms, the indices  $vib$  and  $rot$ , respectively. However, the formal symmetry contains an important unphysical part – all elements from the symmetrization group  $\overline{G}_s = \overline{O}$ . The following diagram shows relations among engaged groups (below are shown labels of irreducible representations required for labelling eigenvectors of  $\hat{\mathcal{H}}$ ):

$$\begin{array}{ccc} \hat{\mathcal{H}} & = & \hat{\mathcal{H}}_{vib} + \hat{\mathcal{H}}_{rot} \\ \downarrow & & \downarrow \quad \downarrow \\ \overline{G}_H & = & \overline{G}_{vib} \times \overline{G}_{rot} \\ \downarrow \quad \downarrow & & \downarrow \quad \downarrow \\ \Gamma_s = 0 \quad \sigma & & \Gamma_v \quad \Gamma_r \end{array} \quad (43)$$

The physical symmetry group should be constructed from the elements of  $G_H$  after excluding all elements of the symmetrization group.

The problem how to construct the appropriate group, if exist, is still not solved.

It turns out, however, that not only symmetry of a Hamiltonian is important but also symmetries of its parts can contribute in properties of a nucleus. We call them partial symmetries.

Let us recall the the Hamiltonian (42), where the vibrational part is represented by vibrational part of the quadrupole Bohr Hamiltonian  $T_{vib;2}$  and a pure octupole  $T_{vib;3}$  term.

The quadrupole part

$$T_{vib;2} = \frac{1}{2} \left\{ \frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} - \frac{1}{\beta^2 \sin(3\gamma)} \frac{\partial}{\partial \gamma} \sin(3\gamma) \frac{\partial}{\partial \gamma} + \beta^2 \right\} \quad (44)$$

is invariant in respect to an intrinsic octahedral group  $\overline{\text{O}}_{quad}$  which acts only on the quadrupole variables.

Observation: the vibrational part of kinetic energy operator of the Bohr Hamiltonian is invariant in respect to an appropriate octahedral group (the rotational part breaks the total symmetry to dihedral one).

A similar form can be obtained for the octupole case. The vibrational kinetic energy operator, for the constant mass tensor, can be written as

$$T_{vib;3} = -\frac{\hbar^2}{2B_3} \sum_{\mu} \frac{\partial}{\partial \alpha_{3\mu}} \frac{\partial}{\partial \alpha_{3\mu}^*} \quad (45)$$

and it is invariant in respect to an intrinsic rotation group  $\overline{\text{SO}}(3)_{oct}$  acting only on octupole degrees of freedom.

Summarizing: the quadrupole kinetic energy is octahedrally invariant (due to restriction on quadrupole variables superimposed by the conditions defining intrinsic frame) and the octupole part of vibrational kinetic energy is invariant in respect to a specific rotation group.

It follows, the potential energy operator in the intrinsic frame, to be a correct physical operator, should have an octahedral symmetry or larger. Similarly the rotational term  $\hat{\mathcal{H}}_{rot}$  should have the appropriate octahedral symmetry or larger.

Assume now a weak (or better NO) coupling between octupole and quadrupole degrees of freedom.

In this case, eigenvectors of this Hamiltonian can be factorized:

$$\Psi_{vib;2;vib;3;JM\nu}(\alpha, \Omega) = \phi_{vib;2}(\alpha_2) \phi_{vib;3}(\alpha_3) R_{JM\nu}(\Omega). \quad (46)$$

Because of symmetry properties of the quadrupole, octupole and the rotational part of the Hamiltonian  $\mathcal{H}$ , each factor in the decomposition (46) belong to an appropriate, irreducible representation of the corresponding symmetry group:

$$\begin{aligned} \overline{\text{O}}_{quad} : \phi_{vib;2}(\alpha_2) &\rightarrow \phi_{\sigma_2 \Gamma_2 a_2}^{vib;2}(\alpha_{20}, \alpha_{22}) \\ \overline{\text{SO}}(3)_{oct} : \phi_{vib;3}(\alpha_3) &\rightarrow \phi_{\sigma_3 \Gamma_3 a_3}^{vib;3}(\{\alpha_{3\mu}\}) \\ \overline{\text{G}}_{rot} : R_{JM\nu}(\Omega) &\rightarrow R_{\sigma_r \Gamma_r a_r}^{JM}(\Omega), \end{aligned} \quad (47)$$

where  $\sigma_k$  describe possible equivalent irreducible representations,  $\Gamma_v = (\Gamma_2, \Gamma_3)$  see the diagram (43) and  $a_k$  label vectors within an irreducible representation.

The factorization (46) and (47) suggests existence of degenerated states due to partial symmetries. However, the problem is more complicated because, at the same time, the physical states have to be invariant in respect to the symmetrization group  $\overline{\text{G}}_s$ . It diminishes a possible degeneration, or destroys it at all. It means, that every case requires careful analysis.

A similar situation is in case of transition among states. The transition operator can be decomposed into tensor parts in respect to required partial symmetries. The matrix elements of these operators can be factorized into the quadrupole,

octupole and the rotational part. It means, one can expect, the partial symmetry quantum numbers can play an important role in special selections rules among states described by these symmetries. The problem is not solved till now in a satisfactory way.

## 6 Conclusions

General notion of intrinsic frame based on the group theoretical approach is a good tool for defining intrinsic variables describing a quantum motion of a nucleus. Usually a transformation to intrinsic frame is not unique. Because of a physical reason, instead of cutting domains of intrinsic variables, one needs to introduce a symmetrization group  $\overline{G}_s$ . It complicates structure of space of states because the space has to be composed of scalar functions in respect to  $\overline{G}_s$ . In addition, a part of intrinsic variables lose their standard transformation properties. However, instead, we can partially separate intrinsic properties of a nucleus like its shape from space position and orientation. The properties which we are able to pull out to an existence depend on a group of motion used to definition of intrinsic frame.

Other words, using of intrinsic variables allow to find additional symmetries of a nucleus which are not seen from the laboratory frame.

A part of important problems related to symmetries in intrinsic variables (mentioned in the paper) are not solved yet in a satisfactory way. Solutions of these problems should allow for more detailed symmetry analysis of nuclear spectra and their transitions. Maybe it will give a tool for explaining different unexpected similarities observed in nuclear spectra.

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