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Structure of Bohr Type Nuclear Collective Spaces – a Few Symmetry Related Problems

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Abstract. Nuclear collective models can be constructed in various ways. One of the most popular and effective method is the prescription given by Bohr and Mottelson. This method leads to constructions of nuclear collective spaces corresponding to the appropriate classical models.

In this paper the general structure of such spaces generated by sets of spherical tensors representing different kinds of nuclear deformations (motions) is revisited. The special features of such spaces constructed in the intrinsic frame are shown.

1 Introduction

Nuclear collective models can be constructed in various ways. One of the most popular and effective method is prescription proposed by Bohr and Mottelson [1–4]. The method can be described as a few steps recipe (to make description more transparent we consider only quadrupole plus octupole motions):

- At the beginning one needs to choose collective variables in the laboratory frame, e.g. quadrupole plus octupole deformation parameters $\{\alpha_{2\mu}^{(lab)}, \alpha_{3\mu}^{(lab)}\}$.
- Using these variables one needs to construct the appropriate classical collective Hamiltonian in the laboratory frame

$$\mathcal{H} = \mathcal{T} + \mathcal{V}(\alpha^{(lab)}), \tag{1}$$

where the kinetic energy is written as

$$\mathcal{T} = \frac{1}{2}\sum_{\lambda\mu} B_\lambda |\dot{\alpha}_{\lambda\mu}^{(lab)}|^2 \tag{2}$$

and the potential energy is given by

$$\mathcal{V} = \mathcal{V}(\alpha^{(lab)}). \tag{3}$$

This collective Hamiltonian is required to be invariant under orthogonal group O(3) acting in the laboratory frame.

Usually, at this moment, the metric tensor of the collective manifold of variables $\alpha^{(lab)}$ is chosen in the form compatible with tensor structure of collective variables:

$$\eta^{lab}_{(\lambda\mu)(\lambda'\mu')} = B_\lambda \sqrt{2\lambda+1}(\lambda\mu\lambda'\mu'|00), \tag{4}$$

where the parameter B_λ denotes constant collective mass for λ-mode, and the expressions $(\lambda_1\mu_1\lambda_2\mu_2|\lambda\mu)$ are standard Clebsch-Gordan coefficients for rotation group. The constant masses only scale the unique metric in the space of laboratory deformation parameters conserving their structure.

- Next step is transformation of these laboratory collective variables and the **classical** Hamiltonian to the so called intrinsic frame – typically to the rotating frame of reference:

$$\alpha_{\lambda\mu} = \sum_{\mu'} D^{\lambda}_{\mu'\mu}(\Omega)\alpha^{(lab)}_{\lambda\mu'} \tag{5}$$

where, in addition

$$\alpha_{2,\pm1} = 0, \quad \alpha_{22} - \alpha_{2,-2} = 0 \tag{6}$$

Note, that the rotation angles are here determined only by quadrupoles, no octupole degrees of freedom are involved.

In the intrinsic frame kinetic energy can be decomposed into vibrational kinetic energy

$$\mathcal{T}_{vib} = \frac{1}{2}\sum_{\lambda\mu} B_\lambda |\dot{\alpha}_{\lambda\mu}|^2 \tag{7}$$

and the rotational kinetic energy

$$\mathcal{T}_{rot} = \frac{1}{2}\sum_{\lambda}\sum_{\nu\nu_1}\sum_{kk_1}\left[B_\lambda \alpha_{\lambda\nu}\alpha_{\lambda\nu_1}(-1)^\lambda \langle\lambda\nu|J_k J_{k_1}|\lambda\nu_1\rangle\right]\omega_k\omega_{k_1}. \tag{8}$$

The dynamic coupling term between vibrations and rotations disappears if $\alpha_{2\pm1} = 0$ and $\alpha_{22} = \alpha_{2,-2}$. The only coupling is given by dependence of inertia parameters on deformation variables.

- The next important step is quantization of constructed classical Hamiltonian – in fact it is not unique procedure. Usually the Podolsky–Pauli prescription for quantization of classical intrinsic (not laboratory) Hamiltonian is used [3]. The most sensitive to the quantisation procedure is kinetic energy.

To define a quantum Hamilton operator one needs also to construct a Hilbert space of states. The problem has also not unique solution. Usually is assumed that the kinetic energy operator is identified with the Laplace-Beltrami operator in intrinsic collective variables:

$$\mathcal{T} \to -\frac{\hbar^2}{2} \sum_{kk'} \frac{1}{\sqrt{|\eta|}} \frac{\partial}{\partial q^k} \sqrt{|\eta|} \eta^{kk'} \frac{\partial}{\partial q^{k'}}, \tag{9}$$

where the metric tensor is given by (4) transformed to intrinsic variables:

$$\eta_{kk'}(q) = \sum_{\lambda\mu,\lambda'\mu'} \frac{\partial \alpha^{(lab)\star}_{\lambda\mu}}{\partial q^k} \frac{\partial \alpha^{(lab)\star}_{\lambda'\mu'}}{\partial q^{k'}} \eta^{lab}_{(\lambda\mu)(\lambda'\mu')}, \tag{10}$$

where q^k are used to label the intrinsic collective variables α and Ω in more uniform form.

The metric tensor (10) determines, in a natural way the volume element in the integral which defines the scalar product in space of square integrable functions of collective variables in the intrinsic frame. The collective Hamiltonian has to be Hermitian in respect to this scalar product. One needs to realize, however, that the same goal one can reach in several ways defining scalar product with additional weight functions.

- The Bohr type procedure leads to intrinsic nuclear collective Hamiltonian which, in fact, is non-physical Hamiltonian – it is not invariant in respect to the symmetrization group. This Hamiltonian has to be "renormalized" in an appropriate way to get physical solutions.

  First one needs to find a set of eigenfunctions of resulting collective Hamiltonian. Next one has to choose within them only physical solutions. The possible physical solutions have to be invariant in respect to the symmetrization group. This procedure is known as symmetrization of collective wave function procedure, e.g. see § 4.1 of [3] and references there in, more recent analysis can be found in the paper [7].

Obviously, there are alternative methods of obtaining differential form of collective quantum Hamiltonian. The Gaussian Overlap Approximation for the Generator Coordinate Method is an example of such derivation.

In this paper the general structure of collective spaces generated by sets of spherical tensors representing different kinds of nuclear deformations (motions) is revisited. The problem became important after suggestions of existence in nuclei higher point symmetries than spherical, axial and tri-axial [8].

We are consider here some, selected special features of spaces constructed in the intrinsic frame. For example, we propose definition of collective variable by construction of appropriate quantum observables. This definition implies a possibility of decomposition of physical properties of intrinsic variables into different classes of quantum motions, e.g. one can separate rotational degrees of

freedom from vibrational variables. Another problem is quality of collective variables. We show some problems with collective surface variables which, in fact, allow for strange, unprobable nuclear shapes. We also revisit the uniqueness problem of transformations from laboratory to intrinsic frame using group orbits notion. The last problem touched in this short article is related to relation between mass parameters and metric in the collective variables space.

2 Collective Variables

There are different ways, not always equivalent, of introducing collective variables. We propose here one more method of defining collective variables based on a set of observables. The method is quite general, but we restrict our considerations to variables related to nuclear surface.

Let q_1, q_2 and q_3 be a set of curvelinear coordinates in R^3. The general equation describing two-dimensional surface in three dimensional space can be written as:

$$q_k = q_k(u, v) \text{ where } k =, 1, 2, 3. \tag{11}$$

where, nearly without loss of generality one can assume that the functions $q_k(u, v)$ are square integrable functions, i.e. $q_k \in L^2(S)$, where the compact subset of variables $S \subset R^2$ parametrizes this nuclear surface.

The space $L^2_\rho(S)$ is a Hilbert space of square integrable functions with the scalar product

$$\langle \psi | \phi \rangle = \int_S dudv\, \rho(u, v)\, \psi(u, v)^\star \phi(u, v), \tag{12}$$

where $\rho(u, v) > 0$ is an appropriate weight function.

In this space one can choose the orthonormal basis $\{e_n(u, v)\}$ having an appropriate physical meaning determined by the set of commuting physical observables $\hat{A}_l$, where $l = 1, 2, \ldots, r$. This choice is arbitrary, however, it should be dependent on the most important either physical or formal features required for collective variables under construction. The orthonormal basis allows to expand the surface functions in the following way:

$$q_k(u, v) = \sum_n C_{n,k}\, e_n(u, v). \tag{13}$$

where the expansion coefficients $C_{n,k}$:

$$C_{n,k} = \int_S dudv\, \rho(u, v)\, e_n(u, v)^\star q_k(u, v) \tag{14}$$

are required collective variables describing the nuclear surface in terms of $\hat{A}_l$.

The multipole collective variables are the most known example of such procedure. In this case the curvelinear coordinates describing nuclear surface are

identified with standard spherical coordinates $\{q_1 = r, q_2 = \theta, q_3 = \phi\}$. The simplest parametrization of nuclear surface can be done by making use: $u = \theta, v = \phi$. The single equation describing shape of a surface can be written as: $r = R(\theta, \phi) \in L^2(SO(2))$. The very natural observables related to this situation are: square of angular momentum operator $\hat{A}_1 = \hat{J}^2$ and its third component $\hat{A}_2 = \hat{J}_z$. The common eigenvectors of both angular momentum operators are given by spherical harmonics $\{e_n(u, v) = Y_{lm}(\theta, \phi)\}$. It leads to known expansion of nuclear surface into spherical harmonics:

$$R(\theta, \phi) = R_0(1 + \sum_{\lambda,\mu} \alpha^{(lab)\star}_{\lambda\mu} Y_{\lambda\mu}(\theta, \phi)). \tag{15}$$

This expansion determines some properties of the laboratory surface collective variables $\alpha^{(lab)}_{\lambda\mu}$ like:

- Reality of surface requires the following relation $(\alpha^{(lab)}_{\lambda\mu})^\star = (-1)^\mu \alpha^{(lab)}_{\lambda,-\mu}$.
- Because the shape of the surface is independent on its orientation, properties of spherical harmonics implies that $\alpha^{(lab)}_{\lambda\mu}$ are covariant components of irreducible spherical tensor of rank λ:

$$\hat{R}(\Omega)\alpha^{(lab)}_{\lambda\mu} = \sum_{\mu'} D^{\lambda}_{\mu'\mu}(\Omega)\alpha^{(lab)}_{\lambda\mu'}, \tag{16}$$

  where $D^{\lambda}_{\mu'\mu}(\Omega)$ are standard Wigner functions of the rotation group SO(3) parametrized by Euler angles $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ and $R(\Omega)$ is rotation operator [9]. In addition, using transformation properties of spherical harmonics in respect to the space inversion operation $\hat{C}_i$ one gets:

$$\hat{C}_i \alpha^{(lab)}_{\lambda\mu} = (-1)^\lambda \alpha^{(lab)}_{\lambda\mu}. \tag{17}$$

For spherical tensors $\xi^{(lab)}_\lambda$ and $\zeta^{(lab)}_\lambda$ one can define scalar product

$$\xi^{(lab)}_\lambda \cdot \zeta^{(lab)}_\lambda = \sum_{\mu\nu} \eta^{(lab)\lambda\mu,\lambda\nu} \xi^{(lab)}_{\lambda\mu} \zeta^{(lab)}_{\lambda\nu} \tag{18}$$

where the metric tensor:

$$\eta^{(lab)\lambda\mu,\lambda'\nu} = \sqrt{2\lambda + 1}(\lambda\mu\lambda'\nu|00) = \delta_{\lambda\lambda'}(-1)^\mu \delta^{-\nu}_{\mu} \tag{19}$$

is determined by the special case of Clebsch–Gordan coefficients $(\lambda_1\mu_1\lambda_2\nu_2|\lambda_2\mu_3)$ which couple both tensor to the total angular momentum 0. This is the only possible (up to multiplicative constant factors) scalar form for spherical tensors. This proves that the metric tensors (4) and (10) are correctly defined.

A very well known example of an important deformation characteristic is the β_λ rotational invariant obtained as scalar product of α_λ deformation variable by itself:

$$\beta_\lambda^2 = \alpha_\lambda^{(lab)} \cdot \alpha_\lambda^{(lab)} = \sum_{\mu\nu} \alpha_{\lambda\mu}^{(lab)} [(-1)^\mu \alpha_{\lambda,-\mu}^{(lab)}] = \sum_\mu |\alpha_{\lambda\mu}^{(lab)}|^2 \in \mathbb{R}. \quad (20)$$

Because β_λ^2 is the only second order rotational scalar this function represents potential energy in the $2\lambda + 1$ dimensional classical harmonic oscillator Hamiltonian:

$$H_{ho} = \sum_\lambda \left[\frac{1}{2} B_\lambda \dot{\alpha}_\lambda^{(lab)} \cdot \dot{\alpha}_\lambda^{(lab)} + \frac{1}{2} B_\lambda \omega_\lambda^2 \beta_\lambda^2 \right]. \quad (21)$$

In more general case than harmonic oscillator collective potential energy can be functions of higher order invariants.

The harmonic oscillator, in reality, usually is able to represent only small vibrations, however, after the quantization the eigenfunctions of quantum harmonic oscillator contain not only small but arbitrarily large deformation variables independently of how small is excitation energy. It leads to the question what kind of shapes are described by the expansion (15). A few figures presented below show some interesting cases which appear while using so defined collective variables. In nuclear physics the expansion (15) usually starts from $\lambda = 2$.

The $\lambda = 0$ term can be replaced by scaling factor which takes into account very small compressibility of nuclear matter.

The $\lambda = 1$ terms are often considered to be space translation of a nucleus described by the nuclear surface (15). This is only approximately true and only for small $\alpha_{1\mu}^{(lab)}$. In Figure 1 there is shown that adding dipole deformation $\alpha_{1\mu}^{(lab)}$ to quadrupole deformation $\alpha_{2\mu}^{(lab)}$ of the nuclear surface one obtains not only a shift of the nucleus but also a change of its shape. This effect is even more pronounced for large quadrupole deformation – the "monster" shapes are presented in Figure 2 Adding to the quadrupole shape a few octupole variables ($\alpha_{20}^{(lab)} = 0.30, \alpha_{22}^{(lab)} = 0.05, \alpha_{32}^{(lab)} = 0.30$) one gets a regular quadrupole+tetrahedral shape (left) but using larger (in nuclei unphysical) deformation parameters ($\alpha_{20}^{(lab)} = 0.90, \alpha_{22}^{(lab)} = 0.90, \alpha_{30}^{(lab)} = 0.30, \alpha_{32}^{(lab)} = 0.30), \alpha_{33}^{(lab)} = 0.05$ one obtains a strange quadrupole+octupole shape (right), see Figure 3 An important observation from the presented figures is, that ranges of collective variables of collective state functions contain also an infinite set of unphysical values which correspond to strange shapes. It implies that more realistic collective state functions should be very small for these values of collective variable to ensure negligible probabilities for contribution of such strange shapes to nuclear collective motion. In addition, it is known, these strange shapes often correspond to rather high liquid drop model energies which is an undesired property.

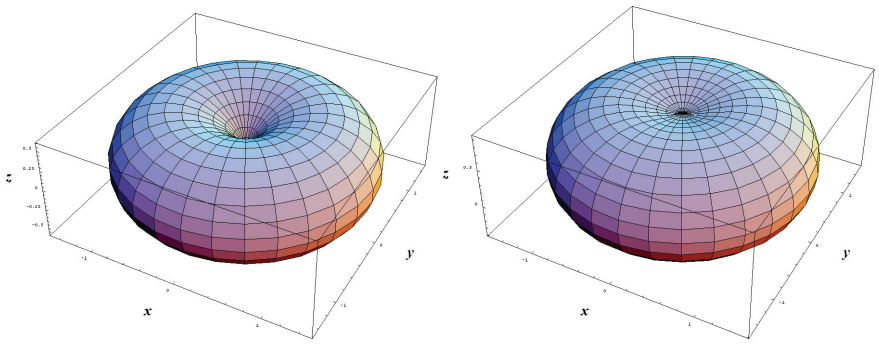

Figure 1. The quadrupole surface $\alpha_{20}^{(lab)} = -1.5$ (left) and the dipole+quadrupole surface $\alpha_{10}^{(lab)} = 1.5$, $\alpha_{20}^{(lab)} = -1.5$ (right)

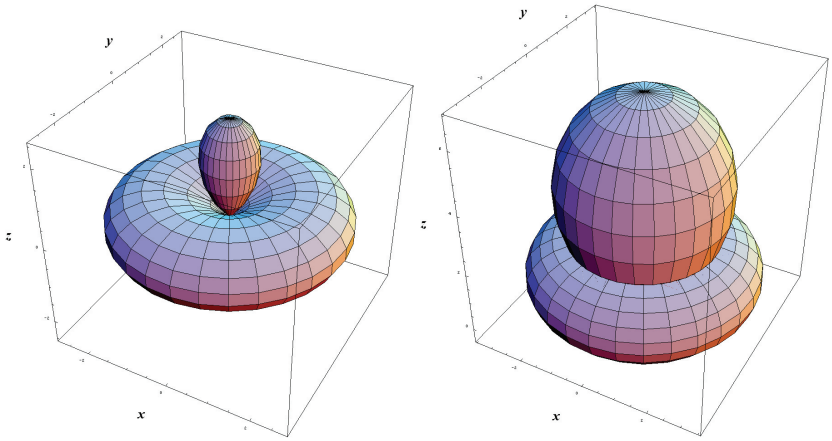

Figure 2. The quadrupole surface $\alpha_{20}^{(lab)} = -5.5$ (left) and the dipole+quadrupole surface $\alpha_{10}^{(lab)} = -9.00$ $\alpha_{20}^{(lab)} = -1.5$ (right)

3 Intrinsic Frame

In previous section we were considered surface collective variables in the laboratory frame. It is easy to notice that whole set of deformation parameters can be collected into subsets representing bodies (surfaces) having the same shape but different space orientations. It suggests that one can reparametrize the laboratory variables to obtain description of nuclear shape and its orientation of the surface in respect to the laboratory frame. This operation is usually called transformation to the rotating intrinsic frame. In the following we summarize main properties of rotating frame. More detailed analysis of transformation from laboratory to intrinsic frame can be found in [10].

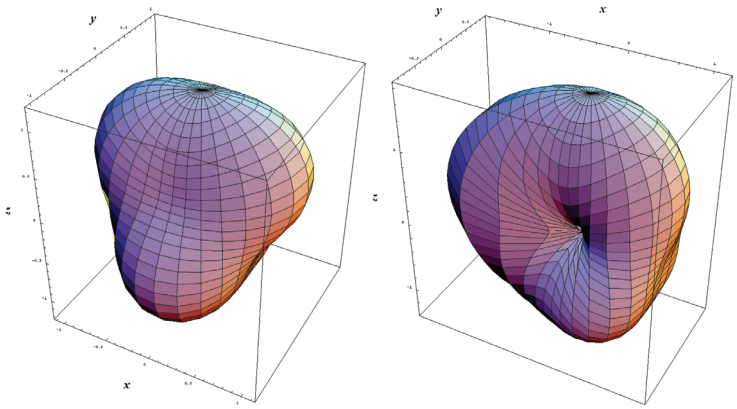

Figure 3. The quadrupole+tetrahedral surface $\alpha_{20}^{(lab)} = 0.3$, $\alpha_{22}^{(lab)} = 0.05$, $\alpha_{32}^{(lab)} = 0.3$ (left) and more complicated quadrupole+octupole surface (right)

Let us consider the collective surface variables in the laboratory frame $\{\alpha_{\lambda\mu}^{(lab)}\}$ and the rotation group SO(3) acting on these variables with the rotation operator $R(\Omega)$, parametrized by the Euler angles Ω. The group parameters $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ are intended to be used as a part of new (intrinsic) collective variables. The transformation formula from the laboratory to intrinsic (rotating) frame can be written as

$$\alpha_\lambda = \hat{R}(\Omega)\alpha_\lambda^{(lab)}, \tag{22}$$

where $\{\alpha_{\lambda\mu}\}$ are redundant intrinsic collective variables – we have $2\lambda + 1$ variables in the laboratory frame, but $(2\lambda + 1) + 3$ intrinsic variables. There are needed 3 additional constraints which couple the Euler angles with the deformation variables $\{\alpha_{\lambda\mu}\}$

$$F_i(\alpha, \Omega) = 0, \qquad \text{where } i = 1, 2, 3. \tag{23}$$

The intrinsic variables α_λ are invariant in respect to the compound rotation $R(h)\mathcal{L}_G(h)$, where $h \in$ SO(3) and $\mathcal{L}_G(h)$ acts on the group manifold of the group SO(3) by left shift operation:

$$\begin{aligned} &R(h)\mathcal{L}_G(h)\alpha_{\lambda\mu} = R(h)\mathcal{L}_G(h)R(\Omega)\alpha_{\lambda\mu}^{(lab)} = \\ &R(h(h^{-1}\Omega))\alpha_{\lambda\mu}^{(lab)} = R(\Omega)\alpha_{\lambda\mu}^{(lab)} = \alpha_{\lambda\mu}. \end{aligned} \tag{24}$$

The operator $R(h)\mathcal{L}_G(h)$ represents a simultaneous rotation of the intrinsic frame in respect to the laboratory one and the corresponding laboratory collective variables by the same angles.

For further purpose we need the notion of intrinsic group. The most general definition is given in the textbook [5]:

Def. For each element g of the group G, one can define a corresponding operator $\bar{g}$ in the group linear space $\mathcal{L}_G$ as:

$$\bar{g}S = Sg, \qquad \text{for all } S \in \mathcal{L}_G. \tag{25}$$

The group formed by the collection of the operators $\overline{g}$ is called the intrinsic group $\overline{\mathrm{G}}$ of the group G.

The most important property of intrinsic groups is that the operation from the 'laboratory' group G and corresponding intrinsic group commute:

$$[G, \overline{\mathrm{G}}] = 0. \tag{26}$$

It means that intrinsic properties defined by an intrinsic group (intrinsic group acts in intrinsic frame) are independent of the corresponding properties in the laboratory frame.

So defined intrinsic groups are anti-isomorphic to their laboratory partners, where the anti-isomorphism is given by:

$$\phi_G : \overline{\mathrm{G}} \to \mathrm{G}, \quad \text{where} \quad \phi_G(\bar{g}) = g \quad \text{and} \quad \phi_G(\bar{g}_1\bar{g}_2) = g_2 g_1. \tag{27}$$

It means that both structure are very similar and most properties of laboratory groups can be directly used for intrinsic groups.

As an example, let us write down the action of the intrinsic rotations $R(\bar{g}_1, \bar{g}_2) \in \overline{\mathrm{SO(3)}}$ in the space of functions of intrinsic variables [11]:

$$\hat{R}(\bar{g}_1, \bar{g}_2) f(\alpha, \Omega) = f(\{\hat{\bar{g}}_1\alpha\}, \Omega\phi_G(\bar{g}_2)^{-1}), \tag{28}$$

where $\bar{g}_1, \bar{g}_2 \in \overline{\mathrm{SO(3)}}$ and

$$\hat{\bar{g}}_1 \alpha_{\lambda\mu} = \sum_{\mu'} D^{\lambda}_{\mu'\mu}(\phi_G(\bar{g}_1)^{-1}) \alpha_{\lambda\mu'}, \tag{29}$$

On the other hand, not all rotations $\bar{g} \in \overline{\mathrm{SO(3)}}$

$$\bar{g}\colon\ (\alpha, \Omega) \to (\alpha', \Omega') \tag{30}$$

are allowed in the intrinsic frame. The rotations $\hat{R}(\bar{g}_1, \bar{g}_2) \in \overline{\mathrm{SO(3)}}$ are allowed if they do not break constraints defining the intrinsic variables

$$F_k(\bar{g}_1\alpha, \Omega\bar{g}_2{}^{-1}) = 0, \text{ where } k = 1, 2, 3, \ldots, r. \tag{31}$$

For instance, for "principal axes Bohr's quadrupole collective variables" the allowed intrinsic rotations $\hat{R}(\bar{g}_1, \bar{g}_2) \in \overline{\mathrm{SO(3)}}$ are those which leave invariant the condition defining the intrinsic frame

$$\hat{R}(\bar{g}_1, \bar{g}_2)\alpha_{2\pm1} = 0 \text{ and } \hat{R}(\bar{g}_1, \bar{g}_2)\alpha_{22} = \hat{R}(\bar{g}_1, \bar{g}_2)\alpha_{2-2}. \tag{32}$$

The conditions (32) do not restrict action of the group $\overline{\mathrm{SO(3)}}$ onto rotational degrees of freedom Ω. It is required property to allow for correct definition of angular momentum operators. The operations $\hat{R}(\bar{g}_1, \bar{g}_2)$ change both an orientation of a nucleus and the intrinsic frame, simultaneously. Rotations of the variables α themselves "move" the frame from its initial position in respect to the body, but, by definition of intrinsic frame this is **not possible**. One can check, that the allowed rotations furnish the octahedral point group $\overline{\mathrm{O}}_\alpha \subset \overline{\mathrm{SO(3)}}_\alpha$ which acts only on the variable α.

In the following, if needed, we will add the extra labels to names of groups which show the variables on which the group acts, e.g. $\overline{\mathrm{D}}_{4h,z;\Omega}$ – the intrinsic group $\overline{\mathrm{D}}_{4h;z}$ has the fourfold symmetry axis in the z directions and acts on the rotational degrees of freedom Ω.

The transformations from laboratory to intrinsic frame are usually not reversible functions. There are possible two scenarios to overcome this undesired property:

(A) One can define the appropriate region of intrinsic collective variables in which the transformation from the laboratory to intrinsic frame is a one-to-one function (it leads often to problems with physical interpretation, e.g. restriction of Euler angles to subrange).

(B) One can work in the whole range of variables, however, one needs to restrict the space of states to the subspace of appropriate periodic functions (symmetrization of eigenstates).

To explain the second idea, which seems to be more physical, it is useful to define a group of intrinsic transformations $\bar{h} \in \overline{\mathrm{G}}_s$:

$$(\alpha, \Omega) \xrightarrow{\bar{h}} (\alpha', \Omega') \tag{33}$$

where $\alpha = \{\alpha_{\lambda\mu}\}$ and which leave invariant the corresponding laboratory coordinates:

$$\alpha^{(lab)}(\alpha', \Omega') = \alpha^{(lab)}(\alpha, \Omega). \tag{34}$$

The group $\overline{\mathrm{G}}_s$ is called the symmetrization group.

The required symmetrization condition for quantum states is given by their invariance in respect to this symmetrization group, i.e., for all $\bar{h} \in \overline{\mathrm{G}}_s$:

$$\bar{h}\Psi(\alpha, \Omega) = \Psi(\alpha, \Omega) \tag{35}$$

In this way we can work with periodic functions which repeat their properties for physically equivalent sets of intrinsic variables.

4 Group Orbits and Uniqueness Problem

It turns out that a good tool for analysis of collective manifold are orbits of the symmetrization group. It is due to the fact, that a symmetrization group decomposes the collective manifold into orbits of physically equivalent points.

Let us denote by $X_{\alpha^{(lab)}}$ and $X_{\alpha\Omega}$ the configuration spaces of laboratory collective variables and intrinsic variables, respectively. To be illustrative, let us consider an example of standard Bohr's shape intrinsic variables $(\alpha_{20}, \alpha_{22})$ equivalent to known Bohr deformation parameters (β, γ).

In this case, the symmetrization group $\overline{\mathrm{O}}$ (inversion dropped) is generated by:

$$\bar{g}_1 = \hat{C}_{2y}, \bar{g}_2 = \hat{C}_{4z}, \bar{g}_3 = \bar{g}(\pi/2, \pi/2, \pi) = \hat{C}_{3\delta}^{-1}, \tag{36}$$

where the Schönflies notation is used, i.e., $\hat{C}_{nw}$ denotes the rotation about the axis w through the angle $2\pi/n$. The direction are chosen as in the appendix D of [6].

The orbit of the collective manifold point $(\beta_0, \gamma_0, \Omega_0)$ defined as:

$$\mathrm{orb}(\overline{\mathrm{O}}; \beta_0, \gamma_0, \Omega_0) = \{\bar{g}(\beta_0, \gamma_0, \Omega_0) : \bar{g} \in \overline{\mathrm{O}}\} \tag{37}$$

consists of 24 elements which give the same laboratory deformation parameters. Here, the action of the intrinsic octahedral group can be written as:

$$\bar{g}\beta = \beta, \quad \bar{g}\gamma \in \{\pm\gamma, \pm(\gamma - k\frac{2\pi}{3})\}, \quad \bar{g}\Omega = \Omega g. \tag{38}$$

The subrange of intrinsic variables in which the transformation from laboratory to intrinsic frame is unique can be constructed as quotient of the $X_{\alpha\Omega}$ manifold and the set of orbits $\mathrm{orb}(\overline{\mathrm{O}})$ of the octahedral group:

$$X^C_{\alpha\Omega} = X_{\alpha\Omega}/\mathrm{orb}(\overline{\mathrm{O}}). \tag{39}$$

The quotient is defined as a family of cosets which are determined by the modulo $\mathrm{orb}(\overline{\mathrm{O}})$ equivalence relation.

Two elements $(\beta', \gamma', \Omega')$ and $(\beta'', \gamma'', \Omega'')$ are called equivalent modulo $\mathrm{orb}(\overline{\mathrm{O}})$ if and only if they belong to the same orbit. It means, they represent the same point in the laboratory collective variable space. More formally:

$$(\beta', \gamma', \Omega') = (\beta'', \gamma'', \Omega'') \ (\mathrm{orb}(\overline{\mathrm{O}})) \tag{40}$$

if there exists (β, γ, Ω), that

$$(\beta', \gamma', \Omega'), (\beta'', \gamma'', \Omega'') \in \mathrm{orb}(\overline{\mathrm{O}}; \beta, \gamma, \Omega). \tag{41}$$

However, in this collective space $X^C_{\alpha\Omega}$ there is a problem with angular momentum definition because in the configuration space $X^C_{\alpha\Omega}$ the Euler angles are restricted to a subrange and not all orientations are available.

A solution of this problem is to join some orbits to recover full range of angles. In this case one can choose the maximal rotation group acting on vibrational degrees of freedom $\overline{\mathrm{O}}_\alpha \times \bar{\mathbf{1}}$ which do not touch the orientation of the nucleus. In this case the orbit has only 6 elements:

$$\mathrm{orb}(\overline{\mathrm{O}}_\alpha \times \bar{\mathbf{1}}; \beta_0, \gamma_0, \Omega_0) = \tag{42}$$

$$\{(\beta_0, \gamma, \Omega_0) : \gamma = \pm\gamma_0, \pm(\gamma_0 - \frac{2\pi}{3}), \pm(\gamma_0 - \frac{4\pi}{3})\} \tag{43}$$

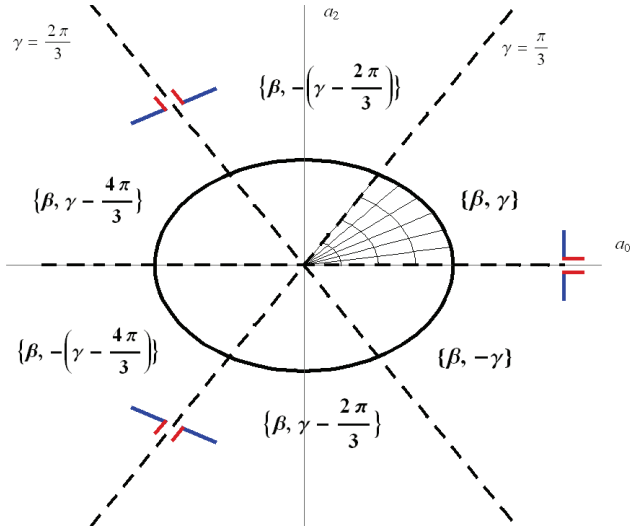

Figure 4. Six equivalent regions, each consist of 4 orbits of the symmetrization group $\overline{\mathsf{O}}$.

and the collective configuration space used in the Bohr model is given by the quotient:

$$X_{\alpha\Omega}^{Bohr} = X_{\alpha\Omega}/\mathrm{orb}(\overline{\mathsf{O}}_\alpha \times \bar{\mathbf{1}}). \tag{44}$$

However, as a result we get again not-invertible $(1 \to 6)$-transformation from the laboratory to intrinsic frame, as shown in Figure 4. It means that, in fact, we only diminished level of non-uniqueness, but still we have to symmetrize states in respect to the symmetrization group. For example, for pure quadrupole case the Bohr Hamiltonian can be rewritten as

$$\hat{\mathcal{H}}_{Bohr} = \hat{\mathcal{H}}_{vib}(\beta,\gamma) + \hat{\mathcal{H}}_{rot}(\Omega) + \hat{\mathcal{H}}_{vr}(\beta,\gamma,\Omega) \tag{45}$$

where the vibrational part is given by

$$\hat{\mathcal{H}}_{vib} = -\frac{\hbar^2}{2B}\left\{\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} - \frac{1}{\beta^2\sin(3\gamma)}\frac{\partial}{\partial\gamma}\sin(3\gamma)\frac{\partial}{\partial\gamma} + \beta^2\right\} + V_B \tag{46}$$

and the rotational term can be expressed as quantum rigid rotator

$$\hat{\mathcal{H}}_{rot} = \frac{1}{2}\sum_{k=1,2,3}\frac{J_k^2}{\mathcal{J}_k}. \tag{47}$$

The resulting coupling term is of the form:

$$\hat{\mathcal{H}}_{vr} = \frac{1}{8\beta^4}\sum_{k=1,2,3}\frac{J_k^2}{\sin^2(\gamma - (2\pi/3)k)} - \hat{\mathcal{H}}_{rot}. \tag{48}$$

This form of the Hamiltonian is more intuitive and better for symmetry analysis.

One can easily check that the vibrational sub-Hamiltonian $\hat{\mathcal{H}}_{vib}(\beta,\gamma)$ has an octahedral symmetry $\overline{\mathrm{O}}_{h;\alpha}$:

$$\mathrm{Sym}(\hat{\mathcal{H}}_{vib}) = \overline{\mathrm{O}}_{h;\alpha}. \tag{49}$$

On the other hand, the rotational sub-Hamiltonian has dihedral symmetry $\overline{\mathrm{D}}_{2h;\Omega}$:

$$\mathrm{Sym}(\hat{\mathcal{H}}_{rot}) = \overline{\mathrm{D}}_{2h;\Omega}. \tag{50}$$

because it is a function of $\overline{\mathrm{D}}_{2h;\Omega}$ invariants. The coupling term has more complicated symmetry:

$$\mathrm{Sym}(\hat{\mathcal{H}}_{vr}) = \overline{\mathrm{O}}_{h;\alpha} \times \overline{\mathrm{D}}_{2h;\Omega} \not\supset \overline{\mathrm{O}}_{h}. \tag{51}$$

The resulting symmetry is the dihedral group $\overline{\mathrm{D}}_{2h}$ acting simultaneously on vibrational and rotational variables. It means that the Bohr Hamiltonian is not invariant in respect to the required symmetrization group. In this context it is unphysical. This requires a symmetrization procedure, as it mentioned earlier in the paper. More detailed description of the problem and its solution is considered in [7].

5 The Mass Parameters Problem

In Bohr type collective models a collective Hamiltonian contains usually two main ingredients: collective kinetic energy determined by mass parameters and potential. The potential is rather well determined physical measurable quantity. On the other hand, mass parameters are not well defined quantum objects but they play an important role in structure of Bohr type collective models.

Mass tensor is usually considered to be metric tensor $B^{Bohr}_{k_1,k_2}(q) \equiv \eta_{kk'}(q)$ in collective variables $\{q^k\}$ manifold. On the other hand, the metric tensor can be written in any arbitrarily chosen variables

$$\eta'_{kk'}(u) = \sum_{k_1,k_2} \frac{\partial q^{k_1}}{\partial u^k} \frac{\partial q^{k_2}}{\partial u^{k'}} B^{Bohr}_{k_1,k_2}(q). \tag{52}$$

However, after the appropriate substitution of new variables in the harmonic oscillator Hamiltonian this new collective Hamiltonian give nothing new but again, another representation of 5-dimensional harmonic oscillator.

These considerations leads to the following question : can we replace $\eta_{kk'}(q)$ by any arbitrary tensor which can be considered as metric tensor, e.g. a cranking mass tensor ?

In general, the answer seems to be negative. The correct tensor metric has to conserve the SO(3) tensor structure of collective variables and at the same time transformation (symmetry) properties. To reach this goal some new collective variables u^k should exist in which the metric tensor determined by cranking

mass tensor $B^{(crank)}_{kk'}(u)$ has to be equal to the metric tensor (52) which is equivalent to the tensor (19) resulting from the structure of rotation group:

$$B^{(crank)}_{kk'}(u) = \sum_{k_1,k_2} \frac{\partial q^{k_1}}{\partial u^k} \frac{\partial q^{k_2}}{\partial u^{k'}} B^{Bohr}_{k_1,k_2}(q). \tag{53}$$

It means, to conserve the appropriate structure of tensor collective variables, as a result, we have to obtain again the kinetic energy with constant mass. On the other hand, this is, in general, not possible – differential geometry shows that not all metric tensors can be transformed to Cartesian metric.

Conclusion: Using of cranking masses $B^{(crank)}_{kk'}$ and at the same time microscopic potential energy V_{coll}, can leads to new collective Hamiltonian which is not equivalent to any collective Hamiltonian with constant masses. However, in many cases, such model also provide a new structure of collective coordinates which properties can be different from their initial features (meaning and transformation properties of variables can be changed). In this way one obtains a new class of collective models which cannot be derived from the five dimensional Harmonic oscillator.

This result shows that in such cases one needs to analyze carefully not only the collective Hamiltonian itself but also structure of collective space which can be not necessarily equivalent to structure of five dimensional Harmonic oscillator space.

There are also other problems concerning structure of Bohr type collective spaces which should be revisited, e.g. how the collective manifold is changing when one omits some of collective variables. It is typical, practical problem in many models considered in literature. Another, important problem is related to more reach structure of collective space represented by intrinsic collective variables than the corresponding laboratory space.

This and other similar problems requires further investigations.

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