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Algebraic Realization of the Pairing-Plus-Quadrupole Model

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Abstract. Correspondence between the $SO(8)$ pairing basis and Elliott's $SU(3)$ basis, that describes collective rotation of nuclear systems with quadrupole deformation is established on the basis of their complementarity to the same LS coupling chain of the shell model number conserving algebra $U(4\Omega) \supset U_L(\Omega) \otimes U_{ST}(4)$. The classification of the basis states for 4 nucleons in the ds shell along both dynamical symmetries is presented as an example and is applied to study the interplay between the pairing and quadrupole interactions in the Hamiltonian of the Pairing-plus-Quadrupole Model, containing both of them as limiting cases. Reasonable choices of values for the interaction parameter strengths of ^{20}Ne are obtained on the basis of comparing the theoretically obtained behavior of the spectrum with the experimental values for the energies of the low laying states.

1 Introduction

It has been understood since the early years of the development of the nuclear structure physics, that the pairing [1] and the quadrupole-quadrupole interactions [2] are the most important short- and long-range interactions that have to be taken into account in the shell-model description of the nuclear systems [3]. Being with different range of action on the nucleons in the valence shells it is quite clear that these interactions actually influence the behavior of the systems in different parts of the shells. The pairing interaction is responsible for the appearance of the pairing gap in the nuclei with only a few nucleons after the closed shells and is therefore associated with the spherical shape of the system. The quadrupole-quadrupole interaction dominates in the nuclei near the mid-shells and so introduces deformation, which is related to the appearance of rotational sequences in the nuclear spectra. Hence in some nuclei each of these interactions could reproduce relatively well the observed behavior of the nuclear system, but in most of the cases the study of the relationship between them is of great importance. This is the main motivation for the development of the Pairing-plus-Quadrupole Model (PQM) [4–6] for the description of the nuclear excitation spectra. It is most successfully done in the framework of the basic shell model representation of the employed interactions, but the applications to real nuclear systems are rather complicated and cumbersome, due to the enormous dimensionality of the basis space in particular for the heavy nuclei. It is already clearly proven that such a problem is easily avoidable by employing a group-theoretical approach [7], which introduces symmetry principles useful in particular for reducing the basis spaces and in the calculation of matrix elements of transitional operators.

In this work we present such an approach by considering the basic interactions of the PQM as invariants of two respective algebras, which reduce the general symmetry of the shell model in a dynamical way. At the same time the two so defined dynamical symmetry chains are both complementary to the Wigner's spin-isospin $SU_{ST}(4)$ symmetry, which establishes the direct connection between these two limiting cases. The latter allows for the investigation of the competing and complementarity features of the pairing and quadrupole interactions in the description of the realistic nuclear systems in the lower shells up to mass numbers $A \sim 100$.

2 The Many-Particle Shell-Model Scheme

The many-particle shell model wave functions are constructed by filling the single-particle orbitals of the valence shells with nucleons, taking into account the Pauli principle [7]. The later constrain is imposed by requiring an antisymmetrization of the total wave function, containing the product of the spatial, spin and isospin parts. In general, this condition and the complementarity of the particle permutation symmetry group and the unitary transformation on the state orbitals [8], allows the use of the simpler case of $U(4\Omega)$ for the classification of the $m-$ particles' wave functions. In $U(4\Omega)$ the number 4 stands for the dimensionality of the spin-isospin space and $\Omega = \sum_i (2l_i + 1)$ is the dimensionality of the considered shell-model valence space, generated by the **LS**-coupling of m nucleons in the $l_1, l_2 \ldots$ orbits of the considered shells.

As a result, the antisymmetric irreducible representations of $U(4\Omega)$ for m particles, labeled by the Young diagrams $\{1^m\}$ can be further partitioned into spin-isospin and spatial parts

$$\begin{array}{ccccc} U(4\Omega) & \supset & U_{ST}(4) & \otimes & U(\Omega) \\ \{1^m\} & & \{\tilde{f}\} & & \{f\} \end{array} \tag{1}$$

under the condition that each of the possible irreps $\{f\} \equiv \{f_1, f_2, f_3, f_4\}$ ($f_1 \geq f_2 \geq f_3 \geq f_4$) and $\{\tilde{f}\} \equiv \{\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{f}_4\}$, where $\tilde{f}_1 \geq \tilde{f}_2 \geq \tilde{f}_3 \geq \tilde{f}_4$ of the two complementary groups $U_{ST}(4)$ and $U(\Omega)$ respectively are conjugated to each other by interchange of the rows and columns in their respective Young tableaux. Consequently both representations can be obtained from each other. Since they are contained in the simple representation $\{1^m\}$ with $m = \sum_i \tilde{f}_i$ shell model wave functions are only labeled with the number of particles m in the valence shell.

2.1 The Spin-Isospin $SU_{ST}(4)$ Symmetry

As suggested by Wigner [9], the invariance of the nuclear forces in respect to rotations in spin and isospin spaces is introduced through the Lie algebra of $SU_{ST}(4)$, whose representations $\{f'\} \equiv \{f_1 - f_2, f_2 - f_3, f_3 - f_4\}$ are conjugated and can be obtained by the representations $\{f\}$ of $U(\Omega)$. It is obvious that the energy of the nuclear states strongly depends on these quantum numbers.

Wigner's supermultiplet model is actually the nuclear **LS**-coupling scheme [7], [10], which employs the reduction

$$\begin{array}{lll} SU_{ST}(4) \supset & SU_S(2) \otimes & SU_T(2), \\ \{\tilde{f}\} & S & T \end{array} \tag{2}$$

that gives the total spin $S = \sum_i s_i$ and isospin $T = \sum_i t_i$ (i enumerates the considered particles) values of the states wave functions, which together with the orbital angular momentum $L = \sum_i l_i$ of $SO_L(3)$ are good quantum numbers in this case. Further, L and S are coupled to total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$. Using the isomorphism of the algebras $SU_{ST}(4) \sim SO_{ST}(6)$, $SU_S(2) \sim SO_S(3)$ and $SU_T(2) \sim SO_T(3)$, another shell-model-reduction chain equivalent to the chain (2) can be identified [11]:

$$\begin{array}{lll} SU_{ST}(6) \supset & SO_S(3) \otimes & SO_T(3) \\ [P_1, P_2, P_3] & S & T \end{array} \tag{3}$$

and used in a conjunction with the spatial reduction of $U(4\Omega)$ in convenient cases, which will be considered further. We should point out, that the $SU_{ST}(4)$ symmetry is broken in a non-dynamical way by the Coulomb and the $l.s$ interaction [10] in the nuclear mean field approach. The role of both of these increases with the nuclear mass number, but the considered **LS**-coupling scheme is still applicable to nuclei up to mass $A \approx 100$.

While in this way the spin S and the isospin T of the system are specified, there is no general rule for obtaining the values of the orbital angular momentum L, contained in the irreps $\{f\}$, since the reduction $U(\Omega) \supset SO_L(3)$ is not a canonical one.

2.2 Rotations and the $SU(3)$ Symmetry

Elliott's $SU(3)$ model [2] provides an elegant and analytically solvable way for obtaining the missing labels in the reduction of the spatial part of the Wigner's $SU_{ST}(4)$ shell model classification to the orbital angular momentum L, by introducing the reduction [12]:

$$\begin{array}{llll} U(\Omega) \supset & & SU(3) \supset & SO(3) \\ \{f\} & \alpha & (\lambda, \mu) & K \;\; L \end{array} \tag{4}$$

where α indicates the multiplicity of the $SU(3)$ representation (λ, μ) in the $U(\Omega)$ representation $\{f\}$. The $SU(3)$ in (4) is generated by the components of the quadrupole operator:

$$Q_\mu = \sum_l \sqrt{8(2l+1)} ({a^\dagger}_{l\frac{1}{2}\frac{1}{2}} \times \tilde{a}_{l\frac{1}{2}\frac{1}{2}})^{(200)}_{(\mu 00)} \tag{5}$$

and the orbital momentum operator

$$L_\mu = \sum_l \sqrt{4l(2l+1)(l+1)/3} ({a^\dagger}_{l\frac{1}{2}\frac{1}{2}} \times \tilde{a}_{l\frac{1}{2}\frac{1}{2}})^{(100)}_{(\mu 00)}, \tag{6}$$

which are presented in a second quantized form. They act in a single valence shell labeled by l and the bracket denotes coupling in the angular momentum, spin and isospin (LST). In addition, the model assumes that the long range residual interaction has a quadrupole character and the Hamiltonian can be written as:

$$H_{rot} = H_0 + \frac{1}{2}\chi Q.Q, \tag{7}$$

where $Q.Q = 4C^2_{SU(3)} - 3L^2$ and the eigenvalue of the second invariant of $SU(3)$ is $C^2_{SU(3)} = \lambda^2 + \lambda\mu + \mu^2 + 3(\lambda + \mu)$. Obviously (7) gives rise to a rotational spectrum of the type $L(L+1)$. The chain (4) is a classical example of dynamical symmetry breaking of the degeneracy within the $U(\Omega)$ or $SU_{ST}(4)$ by the quadrupole interaction. In this way the rotational states are labeled by the quantum numbers of the representations of the algebras in the chain (4):

$$|\Psi_r\rangle \equiv |\{f\}, \alpha(\lambda,\mu)KLM\rangle \equiv |m, \alpha(\lambda,\mu)KLM\rangle. \tag{8}$$

and are obtained in the context of the shell model [10]. Since Elliott's SU(3) model starts with the Wigner's supermultiplet classification, it also breaks down from the spin-orbit term in the nuclear mean field, and causes a considerable rearrangement of the single-particle levels. The model could be applied successfully mainly for nuclei in the ds and fp shells. For treating heavier systems several more refined approaches [10], like the pseudo-spin symmetry [7] have been employed.

2.3 Total (Isoscalar plus Isovector) Pairing Basis

Another way to reduce the multiplicity of the spatial shell-model algebra $U(\Omega)$ to the angular momentum algebra $SO_L(3)$ in the **LS**-coupling scheme is to use the reduction chain:

$$\begin{array}{ccc} U(\Omega) \supset & SO(\Omega) \supset & SO_L(3) \\ \{f\} & [\tilde{\mu}] & \beta\, L \end{array} \tag{9}$$

which can easily be realized using the basic assumption that the fundamental representation $\{1\}_{U(\Omega)}$ is composed by the representations $(l_1, l_2, \ldots l_r)$ of $SO_L(3)$ for nucleons occupying the orbits $l_1, l_2, \ldots l_r$. Then by using the standard methods for the decompositions $U(n) \supset O(n)$ and $O(n) \supset O(n-1)$ [8]

one obtains the values of the angular momentum operators L and their multiplicity β. Hence this reduction explicitly depends on the the $l-$ orbits appropriate for the nucleus under consideration and could be applied in one or several orbits.

Now we turn to another important aspect of this reduction, namely its relation to the description of pairing phenomena in the framework of the shell model algebras. It is proven in [13] that the $SO(\Omega)$, appearing in the $U(\Omega)$ representation space, restricted by the condition to be in a direct product with the $SU_{ST}(4)$ algebra is complementary to the $SO(8)$ algebra. Because of this they both are labeled by the same quantum numbers: $v, [p_1, p_2, p_3]$ or $[\mu] = [\mu_1, \mu_2, \mu_3, \mu_4]$ with $\mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4 \geq 0$, where $v = \sum_i \mu_i$ is the seniority quantum number for $SO(\Omega)$ and $SO(8)$. Using the complementarity of $U(\Omega)$ and $SU_{ST}(4) \sim SO_{ST}(6)$ (1), (3) and $SO(\Omega)$ and $SO(8)$ the basis states in the reduction to the angular momentum subgroup $SO_L(3)$ could be labeled as

$$|\Psi_p\rangle \equiv |\{f\}, v, [p_1, p_2, p_3], \beta LM\rangle \equiv |m, v, [p_1, p_2, p_3], \beta LM\rangle, \qquad (10)$$

where β gives the multiplicity with which the values of the angular momentum L appear in the $SO(8)$ representations $v, [p_1, p_2, p_3]$. On the other hand the $SO(8)$ is the spectrum generating algebra for the isoscalar ($T = 0$) and isovector ($T = 1$) pair creation and annihilation operators within the nuclear shell model:

$$S^\dagger_\mu = \sum_l \beta_l \sqrt{\frac{2l+1}{2}} (a^\dagger{}_{l\frac{1}{2}\frac{1}{2}} \times \tilde{a}^\dagger_{l\frac{1}{2}\frac{1}{2}})^{(010)}_{(0\mu 0)} \qquad (11)$$

and

$$P^\dagger_\mu = \sum_l \beta_l \sqrt{\frac{2l+1}{2}} (a^\dagger{}_{l\frac{1}{2}\frac{1}{2}} \times \tilde{a}^\dagger_{l\frac{1}{2}\frac{1}{2}})^{(001)}_{(00\mu)}, \qquad (12)$$

where $\beta_l = +1$ or -1, and the bracket denotes coupling in the angular momentum, spin and isospin. Obviously these operators can be expressed in terms of the Wigner's $SU_{ST}(4)$ generators. Physically relevant classification of the $SO(8)$ algebra that conserves spin and isospin are realized through three dynamical symmetries [13], [10]:

$$\begin{array}{cccccccc} & \nearrow & SO(6) & & \searrow & & & \\ SO(8) & \rightarrow & SO_S(5)\otimes & SO_T(3) & \rightarrow & SO_S(3)\otimes & SO_T(3) & \qquad (13) \\ & \searrow & SO_T(5)\otimes & SO_S(3) & \nearrow & & & \end{array}$$

which are used to obtain analytical solutions for either isoscalar or isovector pairing interactions in the last two chains or in the first one for both of them with equal strengths. In the latter case, using the relations for the Casimir invariants of the algebras in (13), (2) and (3), the pairing interaction:

$$V_{\text{pair}} = -\frac{1}{2}\mathbf{G}\{(1-x)S^\dagger_\mu.S_\mu + (1+x)P^\dagger_\mu.P_\mu\}, \qquad (14)$$

at $x = 0$ has the eigenvalues [13]:

$$E_{\text{pair}}(m, v, [p], [P], (ST)) = \frac{1}{2}\{-\frac{1}{4}(m-\nu)(4\Omega+12-m-\nu) \\ -[p_1(p_1+4))+p_2(p_2+2)+p_3^2]+[P_1(P_1+4))+P_2(P_2+2)+P_3^2]\}, \quad (15)$$

that do not depend explicitly on L, S and T. This also implies that the interplay between the different types of pairings in the nuclear systems can be investigated.

3 Unifying the Reductions of Shell-Model $U(4\Omega)$ Algebra

Summarizing the reduction of the shell model algebra $U(4\Omega)$ (1) into the spatial and spin-isospin branches and their complementarity and taking into account that the reduction of the spatial part $U(\Omega)$ to the $SO(3)$ algebra of the angular momentum, could be realized in two possible ways through the $SU(3)$ algebra (4) [7] and through the $SO(\Omega) \Leftrightarrow SO(8)$ (9) [13], both complementary to the reduction of the Wigner's supermultiplet model (2) [9], we can unify the considered above chains into a generalized reduction scheme of the type:

$$\begin{array}{cccccc}
\{1^m\} & & & U(4\Omega) & & \\
 & & & \downarrow & & \\
(\tilde{f}\} & [U(\Omega) & & \otimes & U_{ST}(4)] & \{f\} \\
 & \downarrow & \searrow\ \alpha & & \downarrow & \\
[\tilde{\mu}] & [SO(\Omega) & [SU(3) & \otimes & SU_{ST}(4)] & \{f'\} \\
(\Omega-\frac{\nu}{2},[p]) & \Leftrightarrow SO(8) & (\lambda,\mu) & & \sim SO(6) & [P] \\
\beta & \downarrow & \swarrow\ K & & \downarrow & \\
L & [SO_L(3) & & \otimes & SU_S(2)]\otimes SU_T(2) & S,T \\
 & & & \downarrow & \downarrow & \\
J & & & SU_J(2) & \otimes SU_T(2) & T
\end{array} \quad (16)$$

Now we obtain the important result that the spatial subalgebra $U(\Omega)$ of the shell-model algebra $U(4\Omega)$ contains two distinct dynamical symmetries defined by the reduction chains -left branch (9) and middle branch (4). Both of them are complementary to the Wigner's supermultiplet (2) on the right-hand side of the reduction scheme (16). As a result we obtain the residual interaction of the system as:

$$V_{res} = \frac{1}{2}(1-x)\chi Q.Q - \frac{1}{2}(1+x)\mathbf{G}(S_\mu^\dagger.S_\mu + P_\mu^\dagger.P_\mu), \quad (17)$$

where at $x = 1$ we have pure pairing interaction with equal strengths of the isoscalar and isovector terms, and at $x = -1$ the limiting case of pure quadrupole interaction is realized. At $x = 0$ we have both interactions mixed with their respective strengths. This allows us to investigate the influence of these residual interactions on the spectra in real nuclear systems, which will be presented further in the applications. The residual interaction (17) at $x = 0$ defines the Hamiltonian $H = H_0 + V_{res}$ of the Pairing-plus-Quadrupole Model /PQM/ [14].

4 Relation Between the Basis States in the Pairing and Quadrupole Limits

From the above generalized reduction scheme it could be seen that both chains defining the reductions in the spatial part of the generalized scheme (16) determine full-basis sets and could be expressed through each other. Since the microscopic $SU(3)$ model based on the three-dimensional harmonic oscillator has a well-developed theory, including the Wigner-Racah algebra for the calculation of matrix elements [15], [16] in the $SU(3)$ basis and various successful applications in real nuclei [17], we choose to expand the states of the pairing basis (10) with the quantum numbers $\{\nu[p_1, p_2, p_3]\beta\}$ labeled by the set of numbers $\{i\}$ through the set of basis states (8) with the quantum numbers $\{\alpha(\lambda, \mu)K\}$ denoted as the set $\{j\}$:

$$|\Psi_p\rangle_i \equiv |\{f\}, i, LM\rangle = \sum_j C_{ij} |\{f\}, j, LM\rangle. \tag{18}$$

As a result of the dynamical symmetry, the pairing interaction (14) is diagonal in the pairing basis (10) with eigenvalues given by (15). Using its expansion (18) in the $SU(3)$ basis states and the diagonalization procedure for its matrix in the $SU(3)$ basis:

$$\langle\Psi_p|H_{\text{pair}}|\Psi_p\rangle = E_{\text{pair}}(m, i, [P], (ST)) \sum_j C^*_{ji} C_{ij}. \quad \underbrace{{}_j\langle\Psi_r|\Psi_r\rangle_j}_{=1} \tag{19}$$

we obtain numerically the probability $|C_{ij}|^2$ with which the states of the $SU(3)$ basis enter into the expansion of the pairing basis. In this way we actually calculate the transformation brackets between the two chains [18], which is of great use when calculating the matrix elements of different operators in each of the chains. In particular, since we do not have an explicit representation of the pairing basis in terms of the fermion creation and annihilation operators we can use the transformation brackets to calculate different matrix elements in it. This is important for example for the calculation of transition probabilities. Also, this expansion could help evaluate the importance (weight) of the different $SU(3)$ - states, when we need to impose restrictions on the basis because of computational difficulties.

The known relations of the $SU(3)$ labels (λ, μ) and the β, γ shape variables of the geometrical model can be used for the analysis of the deformations of the pairing states, expressed through the respective $SU(3)$ ones.

5 Application of the Algebraic Construction in the Shell-Model Theory for the Case of $m = 4$ (Four) Particles in the ds Shell

After presenting in short the algebraic realization of the dynamical symmetries that appear in the microscopic shell model we would now like to exploit their applications in realistic nuclear systems. We start with the first real test case for the applications of the theory - the ds shell, which is the first one, where both deformation and pairing phenomena play an important role [19], [20]. Our proof of case example presents the simple but complete case of 4 particles in the ds

Table 1. The classification of the states of 4 particles in the ds shell ($\Omega = 6$) according to the reduction scheme (16).

$U(6)$ $\{\tilde{f}\}$	$SO(6)$ $[\mu]$	$SO_p(6)$ $\nu[p]$	$SU(3)$ (λ,μ)	K	$SO(3)$ L	$U_{ST}(4)$ $\{f\}$	$SO_P(6)$ $[P]$	$SU_S(2)\times SU_T(2)$ (ST)
$\{1^4\}_{15}$	[2]	$2[1^3]$	$(1,2)_{15}$	1	$1,2,3$	$\{4\}_{35}$	$[2^3]_{35}$	$(0,0)_1$
								$(1,1)_9$
								$(2,2)_{25}$
$\{21^2\}_{105}$	[31]	$4[21^2]$	$(0,1)_3$	0	1	$\{31\}_{45}$	$[21^2]_{45}$	$(1,0)_3$
			$(2,3)_{42}$	0	$1,3,5$			$(0,1)_3$
				2	$2,3,4$			$(2,1)_{15}$
			$(5,0)_{21}$	0	$1,3,5$			$(1,2)_{15}$
			$(3,1)_{24}$	1	$1,2,3,4$			$(1,1)_9$
	[2]	$2[1^3]$	$(1,2)_{15}$	0	$1,2,3$			
$\{2^2\}_{105}$	[0]	$0[0]$	$(0,4)_{15}$	0	$0,2,4$	$\{2^2\}_{20}$	$[2]_{20}$	$(2,0)_5$
	$[1^2]$	$2[1]$	$(2,0)_6$	0	$0,2$			$(1,1)_9$
	$[2^2]$	$4[2]$	$(4,2)_{60}$	0	$0,2,4,6$			$(0,2)_5$
				2	$2,3,4,5$			$(0,0)_1$
			$(3,1)_{24}$	1	$1,2,3,4$			
$\{31\}_{210}$	[2]	$2[1^3]$	$(1,2)_{15}$	1	$1,2,3$	$\{21^2\}_{15}$	$[1^2]_{15}$	$(1,0)_3$
	$[1^2]$	$2[1]$	$(6,1)_{63}$	1	$1,2,3,4,5,$ $6,7$			$(0,1)_3$
	$[21^2]$	$4[1^2]$	$(4,2)_{60}$	0	$0,2,4,6$			$(1,1)_9$
				2	$2,3,4,5$			
			$(2,3)_{42}$	0	$1,3,5$			
				2	$2,3,4$			
			$(3,1)_{24}$	1	$1,2,3,4$			
			$(2,0)_6$	0	$0,2$			
$\{4\}_{126}$	$[1^4]$	$4[0]$	$(8,0)_{45}$	0	$0,2,4,6,8$	$\{1^4\}_1$	$[0]_1$	$(0,0)_1$
			$(4,2)_{60}$	0	$0,2,4,6$			
				2	$2,3,4,5$			
	[0]	$0[0]$	$(0,4)_{15}$	0	$0,2,4$			
	$[1^2]$	$2[1]$	$(2,0)_6$	0	$0,2$			

shell, which allows us to study the PQM without any truncation of the model space.

In Table 1, using the described reduction rules for the generalized reduction scheme given in (16), we present the classification of 4 particles ($m = 4$) in the ds shell ($\Omega = 6$). Since in this case the space dimension is only 6, we do not employ the complementarity of $SO(\Omega)$ and $SO(8)$ but go down directly from $SO(6) \supset SO(5) \supset SO(3)$ and along the first chain in (16). In the first column of the table the $U(6)$ spatial representations $\{f\}$ conjugated to the ones of $U_{ST}(4)$ are shown with their dimensionality, given by the indexes next to the representation brackets. In the next column their reduction to the respective $SO(\Omega) \equiv SO(6)$ representations $[\mu] \sim v[p]$ is given. In the next, forth column the $SU(3)$ irreps (λ, μ), contained in the $\{f\}$ irreps, are listed with their K, L, M content, which are obtained as well from the $SO(6) \supset SO(5) \supset SO(3)$ reduction. In the last three columns of the table the allowed $U_{ST}(4), SO_P(6)$ and $SU_S(2) \otimes SU_T(2)$ irreps $\{\tilde{f}\}$,$[P]$ and (S, T) representations corresponding to the respective representation $\{f\}$ of $U(6)$ particles in the ds shell, are enumerated. All the possible five cases of 4 protons or 4 neutrons, 3 protons and 1 neutron and 3 neutrons and 1 proton and 2 protons and

Table 2. Decomposition of the pairing states of 2 protons + 2 neutrons in the ds shell in terms of $SU(3)$ basis states.

$\{i\} \equiv \{\|\nu[p][P] >\}$	Energy [MeV]	$\|\{j\} \equiv \{(\lambda,\mu)L, S >\}$	$\|C_{ij}\|^2[\%]$
0_1	-16	$(8,0)0,0$	56.25
$\|0[0][0] >$		$(4,2)0,0$	6.94
		$(0,4)0,0$	34.03
		$(2,0)0,0$	2.78
0_2	-10	$(4,2)0,0$	77.78
$\|2[1][0] >$		$(0,4)0,0$	11.11
		$(2,0)0,0$	11.11
2_1	-10	$(8,0)2,0$	83.39
$\|2[1][0] >$		$(4,2)2,0$	5.76
		$(0,4)2,0$	6.65
		$(2,0)2,0$	4.20
2_2	-10	$(8,0)2,0$	1.46
$\|2[1][0] >$		$(4,2)2,0$	72.50
		$(0,4)2,0$	15.79
		$(2,0)2,0$	10.25
4_1	-10	$(8,0)4,0$	40.86
$\|2[1][0] >$		$(4,2)4,0$	53.81
		$(0,4)4,0$	5.33

2 neutrons, which are defined by the respective values of T in the last column of the table are obtained. For example, since for 4 protons we have $T = 2, T_z = 2$, we should consider from $U(4)$ irrep $\{4\}$ the states with $(S = 2, T = 2)$; from $\{3, 1\}$ with $(S = 1, T = 2)$ and from the $\{2, 2\}$ with $(S = 0, T = 2)$. The case of 3 protons and 1 neutron or 3 neutrons and 1 proton contains all the $U(4)$ representations with $T = 2, T_z = \pm 1$ and $T = 1, T_z = \pm 1$ respectively. The most complex case is the one of 2 protons and 2 neutrons which contains all $U(4)$ representations, listed in the table. From Table 1 the correspondence between the $SO(6)$ representations $v[p]$ and a respective set of $SU(3)$ irreps (λ, μ) is easy to be seen. Obviously, the pairing representations mix all the $SU(3)$ irreps belonging to their respective $U(6)$ representations.

Using the expansion (18) and the eigenvalues (15) in the pairing basis (10) from (19) we obtain the decompositions of the pairing states in terms of the $SU(3)$ basis states. Example of this decomposition of the first few low-lying pairing states into $SU(3)$ basis states is given in Table 2. From this decomposition one can extract information on the intrinsic deformation of the pairing states through the content in percentage of the $SU(3)$ states, which are clearly associated with the $(\beta\gamma)$ variables of the Geometric Collective Model [21]. As could be seen from Table 2 for the yrast states the prolate components $(8, 0)$ and $(4, 2)$ play a dominant role, although for the ground state it is strongly mixed with the oblate $(0, 4)$ state.

Further, we study the interplay between the pairing and quadrupole interactions in (17) for 4 particles of the same kind in the ds shell at changing values of the parameter $-1 \leq x \leq 1$ for fixed reasonable values of their strengths [22]. In Figure 1 we display the energy levels for the first two $J = 0^+$ states, the lowest two of the $J = 2^+$ states and a $J = 4^+$ state.

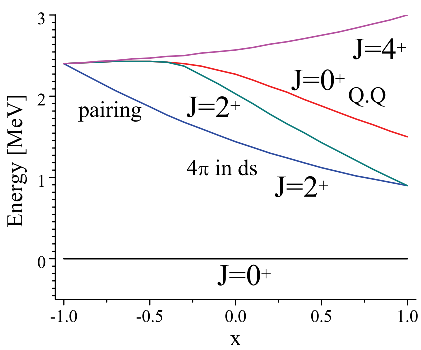

Figure 1. Excitation energies of the PQM (17) for the case of 4 protons (or neutrons) in the ds shell, calculated in the full SU(3) basis at fixed values of $G = 0.4$ MeV and $\chi = 0.1$ MeV, when varying the parameter $-1 \leq x \leq 1$.

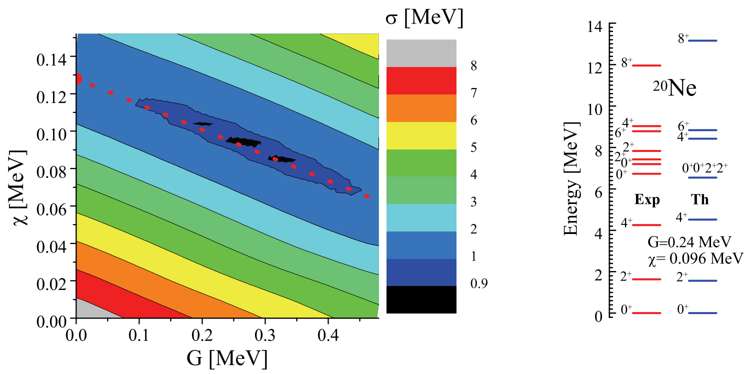

Figure 2. (left) Excitation energies for the PQM interaction for the case of 2 protons and 2 neutrons in the ds shell, calculated in full SU(3) basis; (right) Comparison of the experimental and theoretical spectrum of ^{20}Ne.

At the pairing limit ($x = -1$) a non-degenerate $\nu = 0, J = 0$ state is separated by the degenerated rest of $\nu = 2, J = 0^+, 2^+, 2^+, 4^+$ states by the pairing gap $2\Delta = G\Omega$. Very soon after the pairing limit, for non-vanishing values of χ first the lowest $J = 2^+$ is separated from the rest of the degenerated excited states, then around $x = 0$ all the states degeneracy is removed and the triplet of $\nu = 2, J = 2^+, 0^+, 4^+$ is clearly observed, which reproduces the spectra typical for the quadrupole phonon model [23] as well as in the Interacting Boson Model [24]. In the pure $SU(3)$ limit at $x = 1$ the rotational sequence of $\nu = 2, J = 0^+, 2^+, 4^+$ states in the ground band is recovered, based on the leading $SU(3)$ irrep $(8, 0)$ (see Table 2). As a result, a degeneracy of the two $\nu = 2, J = 2^+$ appears, which is lifted for nonvanishing values of G. The second $\nu = 2, J = 0^+$, based on leading $SU(3)$ irrep $(4, 2)$ is the band head of the excited 0^+. It is obvious that the observed in real systems complicated spectra are best reproduced by taking into account both the pairing and quadrupole-quadrupole interactions.

Based on this on the left side of Figure 2 we present the results of a minimization procedure for the RMS value $\sigma = \sqrt{\sum_i \left(E^i_{\text{Th}} - E^i_{\text{Exp}}\right)^2 / d}$ (per degree of freedom d) with respect to the two parameters G and χ of the residual interaction (17). The black areas in the middle of the figure present the intervals of change of the parameters for which we have the minimal values of σ or the values of the parameters fitted to a set of experimental energies E^i_{Exp} from the observed spectra of a real nuclear system. In the presented case we use the energies of the low-lying states of ^{20}Ne, which has 2 protons and 2 neutrons in the ds valence shell. The red dotted line connects the values of each of the parameters G and χ at their respective limiting cases of pure pairing or pure quadrupole-quadrupole interactions. This line could be assigned as the axis of change of the parameter $-1 \le x \le 1$ used in Figure 1. The regions of the optimal values for the parameters lie on this line and their position in respect to its center could serve as a measure of the influence of each of the terms of the residual interactions on the energy spectra of the considered nucleus.

6 Conclusions

The algebraic structure of the shell-model algebra $U(4\Omega)$ is investigated to obtain its reductions through the microscopic pairing algebra $SO(8)$, containing both isoscalar ($T = 0, S = 1$) and isovector ($T = 1, S = 0$) pairing operators and Elliott's $SU(3)$ algebra that contains the quadrupole Q^2_M and angular momentum operators L^1_M. The two reduction chains appear as two distinct dynamical symmetries of the shell-model algebra, which allows the classification of the basis states of the system along each of them. A relation between these chains is established on the basis of the complementarity to the Wigner's spin-isospin $U_{ST}(4)$ symmetry. This elucidates the algebraic structure of an extended Pairing-plus-Quadrupole Model, realized in the framework of the Elliott $SU(3)$ scheme [7]. The pairing part of the Hamiltonian consists of pp-, nn- and pn-pairing terms. The relationship between the basis states classified in each of the dynamical symmetries is obtained, which allows for the evaluation of the matrix elements of the operators representing each algebra in the reductions. This approach is used to study the combined effects of the quadrupole-quadrupole and pairing interactions on the energy spectra of the nuclear systems.

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