Multipole Transition Operators in Nuclear Collective Models within the GCM+GOA Approximation

A. Góźdź[1], **A. Pędrak**[1], **A. Gusev**[2], **S. Vinitsky** [2]

[1]Institute of Physics, Dept. Math. Phys., University of Maria Curie–Skłodowska, PL-20031 Lublin, Poland
[2]Joint Institute for Nuclear Research, Dubna, Moscow Region, Russia

Abstract. In this paper derivation of the electromagnetic multipole transition operators within the Generator Coordinate Method with the generalized Gaussian Overlap Approximation has been shown. The formulae for the collective transition operators are obtained up to the second order differential operators. A motivation for these calculations is searching of higher point symmetries in nuclei.

1 Introduction

Motivation: searching for higher point symmetries in nuclei.

Nuclear collective models can be constructed in various ways. One of the most popular and effective method is a prescription based on the idea of hydrodynamical model proposed by Bohr and his collaborators many years ago [1, 2, 4]. However, this method allows for construction of an arbitrary set of nuclear collective spaces corresponding to a given classical model. In addition, there are problems to construct the appropriate forms of observables in such spaces – these constructions are often dependent on arbitrarily chosen rules which should be fulfilled by collective observables related to fermionic degrees of freedom. An interesting set of collective observables is the set of operators describing electromagnetic multipole transitions. The electric multipole operators seem to be simpler for construction because they are dependent only on charge density distribution. The magnetic multipole operators are more difficult for derivation because they depend on the electric current density which to some extend is a function of linear and angular momenta operators.

In the literature concerning nuclear collective models there are very few papers related to this problem. The electromagnetic transition operators are often obtained on a phenomenological basis. They are rather a kind of a guess usually based on arbitrarily assumed transformation properties of these operators with a number of free parameters which allow to fit the theoretical transition probabilities to the experimental data.

A. Góźdź, A. Pędrak, A. Gusev, S. Vinitsky

In this paper we show a preliminary work on the multipole electromagnetic transition operators for collective models making use of the Generator Coordinate Method with the generalized Gaussian Overlap Approximation (GCM+GOA).

The usual starting point for calculation of electromagnetic transitions within the microscopic models with nucleonic degrees of freedom is the so called "long–wave" approximation of the transition operators obtained within the field theory for point–like nucleons [3].

In this approximation the electric transition multipole operators of multipolarity l are dependent only on charge density of protons inside of a nucleus

$$M(\text{E}; lm) = \sum_{n=1}^{A} e \left(\frac{1}{2} - t_z(n) \right) r_n^l Y_{lm}(\phi_n, \theta_n), \tag{1}$$

where $t_z(n)$ and (r_n, ϕ_n, θ_n) denote the third component of the isospin quantum number and the spherical coordinates of the n-th nucleon, respectively, e is the electric elementary charge.

The magnetic transition operators are dependent on the nuclear electric current density and the magnetic properties of neutrons moving inside of this nucleus

$$M(\text{M}; lm) = \mu_N \sqrt{l(2l+1)}$$
$$\times \sum_{n=1}^{A} r_n^{l-1} \left(g_s(n) \vec{s}_n + \frac{2g_l(n)}{l+1} \vec{l}_n \right) \cdot \nabla_n \left[r_n^l Y_{lm}(\phi_n, \theta_n) \right], \tag{2}$$

where μ_N is the nuclear magneton, $g_s(n)$ and $g_l(n)$ denote the giromagnetic factors for the n-th nucleon.

On the other hand, there is no clear starting point for derivation of the collective transition operators $Q(\xi; lm)$ from the above microscopic formulae. In this case, there are usually used some analogies, simple models and assumed transformation properties which should be fulfilled for these transition operators [5,6]. For example, the formulae for the collective electric transition operators derived on a base of an interpretation of the average value of the operators (1,2) are calculated for an arbitrarily chosen wave function dependent on both nucleonic and collective variables. These formulae are very often used in applications [7].

One of the most general derivation of the **quadrupole** transition operators based on the algebraic properties of the **quadrupole** collective space is shown in the review paper [8]. The appropriate operators can be obtained in terms of some covariants. For example, the electric transition operators can be written as

$$\hat{M}(E; lm) = \sqrt{\frac{2l+1}{16\pi}} Z e R_0^l \sum_{k=1}^{\frac{l}{2}+1} q_k^{(l)}(\sigma_0, \chi_0) \tau_{k\,lm}(\alpha_2, \sigma_2), \tag{3}$$

where the elementary tensors are expressed in terms of the quadrupole collective variables [8]

$$\sigma_0 = \beta^2, \quad \chi_0 = \beta^3 \cos 3\gamma, \tag{4}$$

$$\sigma_{2m} = -\sqrt{\frac{7}{2}}[\alpha_2 \times \alpha_2]_{2m}, \quad \sigma_{3m} = \sqrt{2}[\alpha_2 \times \sigma_2]_{3m}. \tag{5}$$

In these expressions $\alpha_{2\mu}$ denotes the quadrupole variables introduced in [1] and (β, γ) are the corresponding polar coordinates in the intrinsic frame.

The special case of the quadrupole transition operator is then dependent on the collective variables (β, γ) in the following way

$$\hat{M}(E; 2m) = \sqrt{\frac{5}{16\pi}} Z e R_0^2 \left[D_{m0}^2(\Omega)^\star \left(q_1 \beta \cos \gamma + q_2 \beta^2 \cos 2\gamma \right) \right.$$
$$\left. + \left(D_{m2}^2(\Omega)^\star + D_{m,-2}^2(\Omega)^\star \right) \frac{1}{\sqrt{2}} \left(q_1 \beta \sin \gamma - q_2 \beta^2 \sin 2\gamma \right) \right], \tag{6}$$

where the Euler angles $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ determine the space orientation of a nucleus and $D_{m,m'}^l(\Omega)$ are the Wigner functions for the rotation group.

The formulae for the magnetic transition operators are not "static" and they are dependent on the angular momentum operator $\hat{L}_{1\mu}$:

$$\hat{M}(M; lm) = \sqrt{\frac{2l+1}{4\pi}} \frac{Z}{A} \mu_N R_0^{l-1} \sum_{\kappa=l-1}^{l+1} \left[\Lambda_\kappa \times \frac{1}{\hbar} \hat{L}_1 \right]_{lm}, \tag{7}$$

where the appropriate factors are given by

$$\Lambda_{\kappa\mu}(\alpha_2) = \sum_{k=1}^{k_\kappa} g_k^{(\kappa)}(\sigma_0, \chi_0) \tau_{k\ \kappa\mu}(\alpha_2, \sigma_2, \chi_3). \tag{8}$$

In this case, the simplest and the most important is the dipole magnetic transition operator, which can be written as

$$\hat{M}(M1; u) = \sqrt{\frac{3}{4\pi}} \frac{Z}{A} \frac{\mu_N}{\hbar} g_u(\beta, \gamma) \hat{L}_u, \quad \text{where} \ u = x, y, z, \tag{9}$$

and where $\hat{L}_u$ denotes the Cartesian components of the angular momentum operator and the giromagnetic factors are given by

$$g_u(\beta, \gamma) = g_0^{(0)} - \sqrt{\frac{2}{5}} g_1^{(2)} \beta \cos \gamma_u - g_2^{(2)} \beta^2 \cos 2\gamma_u. \tag{10}$$

As one can see, the above formulae based on the Gordan–Hilbert Finiteness Theorem [9] are dependent on a set of arbitrary constants, even the scaling factor in front of the operator is a free parameter. Usually this scaling factor is

chosen to have a correspondence with classical expression for the electromagnetic moments for arbitrarily chosen simplified model of the charge and current densities in the nucleus. In addition, there are technical difficulties to construct the appropriate tensors for higher multipolarities than 2.

In the following we show derivation of the multipole transition operators for the collective model based on the Generator Coordinate Method with the extended Gaussian Overlap Approximation [16]. One promising example of this derivation was done in the paper [10] for axially symmetric components of the electric transition operators with even multipolarity 0, 2 and 4.

The motivation for derivation of all components of the collective transition operators on a microscopic basis is searching for higher point symmetries in nuclei within the TETRANUC collaboration, as for example the tetrahedral symmetry predicted more than ten years ago [11].

2 GCM+GOA

The Generator Coordinate Method belongs to a set of projection methods. Usually it is presented as the method based on the trial function of the following integral form

$$|\Psi\rangle = \int dq \, f(q) \, |q\rangle \,, \tag{11}$$

where $q = (q^1, q^2, \ldots, q^s)$ is a set of collective variables and $|q\rangle$ is the generating function dependent on nucleonic degrees of freedom parametrized by these collective variables, for short review see e.g. [12]. The Griffin-Hill-Wheeler equation [13,14] obtained within this framework allows for calculation of eigenstates and eigenvectors in the restricted space spanned by the family of trial functions (11). More formally the GCM-space should be defined as the minimal, closed space of nuclear states spanned by the generating function $|q\rangle$, i.e. the family of functions $|q\rangle$, where $q = (q^1, q^2, \ldots, q^s)$ runs over the whole range available for these collective variables. However, in practice using of trial functions (11) is in most cases sufficient.

One deficiency of GCM is that the GCM method does not allow to transform the operators acting in the state space of nucleons into the operators acting in the corresponding collective space. To be honest, there exist the formal transformation invented by Brink and Weiguny [15] which transform the subspace of nuclear states spanned by the generating function to the corresponding collective subspace, however, it is too complicated for practical use.

Instead, more useful is using of the Gaussian Overlap Approximation (GOA) which allows to obtained the approximate transformation from the nucleon space to collective space in terms of differential operators. In this paper the extended GOA is used [16]. This form of GOA extends the range of applicability of this approximation.

The main assumption of this approach is that there exists a set of coordinates $\{y^k\}$ to which the collective variables $\{q^k\}$ can be transformed and in which

the overlap function of the generating functions is of the Gaussian shape. More precisely, one can approximate the overlap of the generating functions $\langle q|q'\rangle$ as follows:

$$\langle q|q'\rangle \approx \langle q|q'\rangle_G = \exp\left\{-\frac{1}{2}\sum_{k=1}^{s}(\Gamma^k(q,q'))^2\right\}, \tag{12}$$

where the required coordinates $\{y^k\}$ can be defined as

$$y^k = \Gamma^k(q,q_0), \tag{13}$$

$$\Gamma^k(q,\tilde{q}) + \Gamma^k(\tilde{q},q') = \Gamma^k(q,q'), \tag{14}$$

$$\Gamma^k(q,q') + \Gamma^k(q',q) = 0. \tag{15}$$

q_0 is here an arbitrary fixed point in the collective space. To show how general is this form we write an example of realization of the $\Gamma^k(q,q_0)$ functions

$$\Gamma^k(q,q_0) = \int_{C(q_0,q)}\sum_{\nu=1}^{s} u_\nu^k(q)dq^\nu, \tag{16}$$

where $u_\nu^k(q)$ are **arbitrary functions** and $C(q_0,q)$ is the curve joining two points q_0 and q.

Sometimes there is useful even more general Gaussian approximation

$$\langle q|q'\rangle \approx \langle q|q'\rangle_G = \exp\left(i\gamma(q,q')\right)\exp\left\{-\frac{1}{2}\sum_{k=1}^{s}\left(\Gamma^k(q,q')\right)^2\right\}. \tag{17}$$

However, in the following the phase γ is assumed to be 0.

The Gaussian Overlap Approximation was invented for the Hermitian operators.

Let us assume the operator $\hat{A}$ acting on the nucleonic degrees of freedom is Hermitian. Using of the trial functions (11) a general expression for the matrix elements of this operator can be written as:

$$\langle\Psi_2|\hat{A}|\Psi_1\rangle = \int dq\,dq'\,f_2(q)^\star h(q,q')\,\langle q|q'\rangle\,f_1(q')$$

$$\approx \int dq\,dq'\,f_2(q)^\star h(q,q')\,\langle q|q'\rangle_G\,f_1(q'), \tag{18}$$

where the reduced overlap function of the operator $\hat{A}$ is

$$h(q,q') = \frac{\langle q|\hat{A}|q'\rangle}{\langle q|q'\rangle}. \tag{19}$$

In the above expression the only approximation is replacement of the true overlap function by the generalized Gaussian form (12). This replacement allows

7

for an expansion (to the second order, or higher) in terms of polynomials of differential operators [16]:

$$\langle \Psi_2| \hat{A} |\Psi_1\rangle \approx \int dq \sqrt{|g|}\, \phi_2(q)^\star (\hat{V}_A + \hat{F}_A + \hat{T}_A + \dots)\phi_1(q'), \qquad (20)$$

where the metric tensor and its determinant can be written as:

$$g_{\mu\nu} = -\left(\frac{\partial}{\partial s^\mu} \frac{\partial}{\partial s^\nu} \left\langle q + \frac{s}{2} \Big| q' - \frac{s}{2} \right\rangle \right)_{s=0}, \qquad (21)$$

$$|g| = \det(g_{\mu\nu}). \qquad (22)$$

The operators $\hat{V}_A$, $\hat{F}_A$ and $\hat{T}_A$ are differential operators of 0, 1 and 2 order, respectively. The dots denote the higher order terms. In case when $\hat{A}$ is a Hamiltonian the first three terms correspond to the potential energy, the asymmetry operator and the kinetic energy, respectively.

The corresponding collective functions ϕ are in a complicated manner determined by the weight functions f contained in the trial functions (11)

$$\phi(q) = \int d\xi\, F(f(\xi), \langle q|\xi\rangle_G). \qquad (23)$$

It means that the representation of the operator $\hat{A}$ in the collective space up to the second order can be written as:

$$\hat{A} \to \hat{\mathcal{A}} \approx \hat{V}_A + \hat{F}_A + \hat{T}_A, \qquad (24)$$

where the 0–order approximation in the number of differential operators is

$$\hat{V}_A(q) = \langle q| \hat{A} |q\rangle - \epsilon_0(q), \qquad (25)$$

$$\epsilon_0(q) = \frac{1}{2} g^{\mu\nu} \mathrm{Re} \left[\frac{\Delta}{\Delta q^\mu} \left(\frac{\Delta h(a, a')}{\Delta a^\nu} \right)_q - \left(\frac{\Delta}{\Delta a^\mu} \frac{\Delta h(a, a')}{\Delta a^\nu} \right)_q \right], \qquad (26)$$

where the $\frac{\Delta}{\Delta x^\mu}$ denotes the appropriate covariant derivative in respect to the variable x^μ, e.g. the covariant derivative of the covariant tensor $C_\nu(x)$ is defined as

$$\frac{\Delta C_\nu(x)}{\Delta x^\mu} = \frac{\partial C_\nu(x)}{\partial x^\mu} - \Gamma^\kappa_{\nu\mu} C_\kappa(x), \qquad (27)$$

where $\Gamma^\kappa_{\nu\mu}$ are Christoffel symbols of the second kind. The index q means that the action of the operator is calculated at the point $a = a' = q$ of the collective manifold:

$$\left(\hat{\zeta}(a, a') \right)_q \equiv \left(\hat{\zeta}(a, a') \right)_{a=a'=q}. \qquad (28)$$

The 1st–order approximation in derivatives is given by

$$\hat{F}_A = -\frac{i}{2} g^{\mu\nu} \text{Im} \left[\frac{\Delta}{\Delta q^{\mu}} \left(\frac{\Delta h(a, a')}{\Delta a^{\nu}} \right)_q \right]$$

$$+ (-i) g^{\mu\nu} \text{Im} \left[\left(\frac{\Delta h(a, a')}{\Delta a^{\nu}} \right)_q \right] \frac{\partial}{\partial q^{\nu}} \tag{29}$$

and the 2nd–order differential operator is of the form of the Laplace–Beltrami operator:

$$\hat{T}_A = -\frac{1}{2\sqrt{|g|}} \frac{\partial}{\partial q^{\mu}} \sqrt{|g|} (\mathcal{M}^{-1}(q))^{\mu\nu} \frac{\partial}{\partial q^{\nu}}, \tag{30}$$

where the the second order tensor (for $\hat{A} = \hat{H}$, where $\hat{H}$ is a Hamiltonian, this tensor is interpreted as the inverse of the mass tensor) contains a physical meaning of this term

$$(\mathcal{M}^{-1}(q))^{\mu\nu} = \frac{1}{2} g^{\mu\rho} \text{Re} \left[\left(\frac{\Delta}{\Delta a^{\rho}} \frac{\Delta h(a, a')}{\Delta a'^{\sigma}} \right)_q - \left(\frac{\Delta}{\Delta a^{\rho}} \frac{\Delta h(a, a')}{\Delta a^{\sigma}} \right)_q \right] g^{\sigma\nu}. \tag{31}$$

In this way we have obtained an approximate image of the fermionic operators in a collective space.

3 Collective Transition Operators

The transformation from the nucleonic space of states to the corresponding (approximate) collective space, obtained above, was done for Hermitian operators. The electromagnetic transition operators $\hat{M}(\xi = \text{E}, lm)$ and $\hat{M}(\xi = \text{M}, lm)$ are not Hermitian. In this case the simplest solution which allows to use the GOA approximation is to express the transition operators in terms of Hermitian forms:

$$\hat{M}_+(\xi, lm) = \frac{1}{2} (\hat{M}(\xi, lm) + \hat{M}(\xi, lm)^{\dagger})$$

$$= \frac{1}{2} (\hat{M}(\xi, lm) + (-1)^m \hat{M}(\xi, l, -m)), \tag{32}$$

$$\hat{M}_-(\xi, lm) = \frac{-i}{2} (\hat{M}(\xi, lm) - \hat{M}(\xi, lm)^{\dagger})$$

$$= \frac{1}{2} (\hat{M}(\xi, lm) + (-1)^{m+1} \hat{M}(\xi, l, -m)), \tag{33}$$

where $\xi = E, M$ labels the electric and magnetic transitions, respectively.

The full electromagnetic transition operators can be expressed by these Hermitian operators as:

$$\hat{M}(\xi, lm) = \hat{M}_+(\xi, lm) + i\hat{M}_-(\xi, lm). \tag{34}$$

An important remark is that $\hat{M}_{\pm}$ are not the real and the imaginary part of the corresponding transition operator – the matrix elements of the operators $\hat{M}_{\pm}$ are

in general complex numbers. It is important for evaluation of appropriate matrix elements in the formulae (25), (29) and (30).

To apply the GOA approximation to the transition operators one needs to derive the collective representations of the operators $\hat{M}_{\pm}(\xi, lm)$. For this purpose one needs to use the reduced overlaps for the operators $\hat{M}_{\pm}(\xi, lm)$

$$
\begin{aligned}
h(lm; q, q') &= \frac{\langle q | \hat{M}(\xi, lm) | q' \rangle}{\langle q | q' \rangle} \\
&= \frac{\langle q | \hat{M}_+(\xi, lm) | q' \rangle}{\langle q | q' \rangle} + i \frac{\langle q | \hat{M}_-(\xi, lm) | q' \rangle}{\langle q | q' \rangle},
\end{aligned} \tag{35}
$$

i.e we denote the appropriate reduced overlaps as:

$$
h(lm; q, q') = h_+(lm; q, q') + i h_-(lm; q, q'). \tag{36}
$$

Making use of the reduced overlaps $h_{\pm}(lm; q, q')$ allow to obtain the collective representation $\hat{Q}(\xi; lm)$ of $\hat{M}(\xi; lm)$, up to the second order, within GCM+GOA approach as follows

$$
\begin{aligned}
\hat{M}(\xi; lm) &\rightarrow \hat{Q}(\xi; lm) \\
&\approx (\hat{V}_+ + i\hat{V}_-) + (\hat{F}_+ + i\hat{F}_-) + (\hat{T}_+ + i\hat{T}_-).
\end{aligned} \tag{37}
$$

In most cases one can choose the generating function which fulfil the following condition for the overlap

$$
\text{Im} \langle q | \frac{\partial}{\partial q} | q \rangle = 0. \tag{38}
$$

Usually this can be achieved by the appropriate choice of phases in the single particle functions. In general, this condition is not crucial for the GOA approximation but it simplifies calculations significantly.

In this case the metric tensor can be expressed as:

$$
g_{\mu\nu} = \langle q | \frac{\overleftarrow{\partial}}{\partial q^{\mu}} \frac{\overrightarrow{\partial}}{\partial q^{\nu}} | q \rangle. \tag{39}
$$

The zero order GOA approximation (25) contains two terms. The first one is the simplest matrix element which do not include any derivative

$$
h_{\pm}(lm; q, q') = \langle q | \hat{M}_{\pm}(\xi, lm) | q \rangle. \tag{40}
$$

The required first order derivatives of the reduced overlaps needed for the 0 and 1st order approximation can be written as the following matrix elements

$$
\left(\frac{\partial h_{\pm}(lm; a, a')}{\partial a^{\nu}} \right)_{a=a'=q} = \langle q | \frac{\overleftarrow{\partial}}{\partial q^{\mu}} \hat{M}_{\pm}(\xi, lm) | q \rangle \tag{41}
$$

and the covariant derivative of (41) can be expressed in the following form

$$
\frac{\Delta}{\Delta q^\mu} \left(\frac{\partial h_\pm(lm; a, a')}{\partial a^\nu} \right)_{a=a'=q} = \langle q| \frac{\overleftarrow{\partial}^2}{\partial q^\nu \partial q^\mu} \hat{M}_\pm(\xi, lm) |q\rangle
$$
$$
+ \langle q| \frac{\overleftarrow{\partial}}{\partial q^\nu} \hat{M}_\pm(\xi, lm) \frac{\overrightarrow{\partial}}{\partial q^\mu} |q\rangle - \Gamma^k_{\mu\nu}(q) \langle q| \frac{\overleftarrow{\partial}}{\partial q^\mu} \hat{M}_\pm(\xi, lm) |q\rangle . \quad (42)
$$

Note: the above expression can be equal to zero even if the partial derivatives are different from zero (this happens for example for typical Bohr like Hamiltonians).

The other required matrix elements are:

$$
\left(\frac{\Delta}{\Delta a^\mu} \frac{\Delta h_\pm(lm; a, a')}{\Delta a^\nu} \right)_{a=a'=q} = \langle q| \frac{\overleftarrow{\partial}^2}{\partial q^\nu \partial q^\mu} \hat{M}_\pm(\xi, lm) |q\rangle
$$
$$
- \langle q| \frac{\overleftarrow{\partial}^2}{\partial q^\nu \partial q^\mu} |q\rangle \langle q| \hat{M}_\pm(\xi, lm) |q\rangle - \Gamma^k_{\mu\nu}(q) \langle q| \frac{\overleftarrow{\partial}}{\partial q^\mu} \hat{M}_\pm(\xi, lm) |q\rangle \quad (43)
$$

and

$$
\left(\frac{\Delta}{\Delta a^\mu} \frac{\Delta h_\pm(lm; a, a')}{\Delta a'^\nu} \right)_{a=a'=q} =
$$
$$
+ \langle q| \frac{\overleftarrow{\partial}}{\partial q^\mu} \hat{M}_\pm(\xi, lm) \frac{\overrightarrow{\partial}}{\partial q^\nu} |q\rangle - \langle q| \frac{\overleftarrow{\partial}^2}{\partial q^\mu \partial q^\nu} |q\rangle \langle q| \hat{M}_\pm(\xi, lm) |q\rangle . \quad (44)
$$

In this way we have obtained the formulae for collective transition operators derived directly from the single particle transition operators. It is clear that these operators can be used for the collective models obtained by the GCM+GOA method. The GCM+GOA leads to a special form of the collective space. It is not clear if one can use the same form of the electromagnetic transition operators for collective models derived on different basis than this Gaussian approximation. They can lead to different form of the corresponding collective space. This is an open question which requires further investigation.

Acknowledgement

A. Góźdź, A. Pędrak, A. Gusev, S. Vinitsky

References

[1] A. Bohr, *Mat. Fys. Medd. Dan. Vid. Selsk.* **26** No.14 (1952); A. Bohr and B.R. Mottelson, *Mat. Fys. Medd. Dan. Vid. Selsk.* **27** No.16 (1953).

[2] A. Bohr and B.R. Mottelson, "*Nuclear Structure*", vol. II (W.A. Benjamin Inc., 1975).

[3] A. Bohr and B.R. Mottelson, "*Nuclear Structure*", vol. I (W.A. Benjamin Inc., 1969).

[4] M.K. Pal, "*Theory of Nuclear Structure*" (Affiliated East-West Press, New Dehli, 1981).

[5] S.A. Williams, *Phys. Rev*. **125** (1962) 340.

[6] L.L. Foldy, *Phys. Rev*. **92** (1953) 178.

[7] J.M. Eisenberg and W. Greiner, "*Nuclear theory*", Vol.1 (North-Holland, 1970).

[8] L. Próchniak, S.G. Rohoziński *J. Phys. G* **36** (2009) 123101,

[9] H.W. Turnbull, "*The Theory of Determinants, Matrices and Invariants*" (London, Blackie, 1946).

[10] B. Nerlo-Pomorska et al., *Nuclear Physics* **A462** (1987) 252.

[11] J. Dudek, A. Góźdź, N. Schunck and M. Miśkiewicz, *Phys. Rev. Lett*. **88** (2002) 252502-1.

[12] P. Ring, P. Schuck, "*The Nuclear Many-Body Problem*" (Springer Verlag, New York, 1980).

[13] D.L. Hill, J.A. Wheeler, *Phys. Rev*. **89** (1953) 112.

[14] J.J. Griffin, J.A. Wheeler, *Phys. Rev*. **108** (1957) 311.

[15] D.M. Brink, A. Weiguny, *Nucl. Phys*. **A120** (1968) 59.

[16] A. Góźdź, *Phys. Lett*. **152B** (1985) 281