NUCLEAR THEORY, Vol. 33 (2014)
eds. A.I. Georgieva, N. Minkov, Heron Press, Sofia

Do New Quantum Statistics Exist in Nature?

T.D. Palev

Institute for Nuclear Research and Nuclear Energy, 1784 Sofia, Bulgaria

Abstract. We recall the definition of a Wigner quantum system (WQS). In the frame of this more general to the canonical approach we remind the definition of A-, (resp B-, C- and D-) (super)statistics and outline their relation to the classes A-, (resp B-, C- and D-) of Lie (super)algebras. As an illustration we describe shortly some of the physical properties of an A-oscillator and A-superoscillator. Both of them fall into the cathegory of finite quantum systems and have quite unusual properties.

1 Introduction. Wigner Quantum Systems

In [1] we have shown that the canonical quantum statistics, i.e., the Bose and the Fermi statistics, can be considerably generalized if one abandons the requirement that the commutators or the anticommutators between the fields in quantum field theory (QFT) or the commutators between the position and the momentum operators in quantum mechanics (KM) to be $c-$numbers.

The purpose of the present paper is to outline shortly where the idea (we refer to it as to *the main idea*) for a possible generalization of quantum statistics came from, and to list as an illustration some of the unusual predictions of the new statistics, mainly of $A-$(super)statistics. Throughout we skip the proofs of most of the propositions, which will considerably simplify the exposition.

Chronologically Wigner was the first, who, back in 1950, made a decisive step towards generalization of quantum statistics in quantum mechanics [2]. In order to indicate where this possibility came from, we first recall the axioms of quantum mechanics(QM) as given by Dirac [3]

A1. To every state there corresponds a normed to 1 wave function Ψ.
A2. To every physical observable L there corresponds a selfadjoint operator $\hat{L}$.
A3. The observable L can take only those values which are eigenvalues of $\hat{L}$.
A4. The expectation value L_Ψ of L in the state Ψ is $\langle\Psi|\hat{L}|\Psi\rangle$.
A5. The Heisenberg equations in the Heisenberg picture hold:

$$\dot{\hat{p}}_k = -\frac{\mathrm{i}}{\hbar}[\hat{p}_k, H], \quad \dot{\hat{q}}_k = -\frac{\mathrm{i}}{\hbar}[\hat{q}_k, H]. \tag{1}$$

A6. The canonical commutation relations (CCR's) hold:

$$[\hat{q}_j, \hat{p}_k] = \mathrm{i}\hbar\delta_{jk}, \quad [\hat{q}_j, \hat{q}_k] = [\hat{p}_j, \hat{p}_k] = 0. \tag{2}$$

A key outcome from the above axioms and especially from A5 and A6 is the following consequences

C1. The (operator) equations of motion (the Hamiltonian equations) hold too,

$$\dot{\hat{p}}_k = -\frac{\partial H}{\partial \hat{q}_k}, \quad \dot{\hat{q}}_k = \frac{\partial H}{\partial \hat{p}_k}. \tag{3}$$

C2. From the CCR A6 and the equations of motion C1 one derives the Heisenberg equations C2.

At this place let us ask a question: can some of the axioms A1-A6 from above be removed or modified or replaced or be weakened somehow?

Clearly no one from the first four axioms can be removed without changing the very essence of QM. For the same reason one cannot touch A5, since in the Schrodinger representation this axiom leads to the Schrodinger equation! It is also clear that the equations of motion (C1) have to hold in any case, since they are responsible for the correct classical limit.

What is left? It remains to analyze axiom A6. The first impression is that the axiom A6 has to remain too, since then also the Hamiltonian equations C1 hold. The more precise statement is however that the CCR's are sufficient in order C1 to be fulfilled. Are the CCR's also necessary? This as a question asked by Wigner [2]. And the answer was "NO"! The CCR's are not necessary in order the equation of motion C1 to hold! Wigner proved this [2] on an example of one-dimensional harmonic oscillator. Unknown were the position and the momentum operators. Parallel to the canonical solution Wigner found infinitely many other solutions, which also satisfied A5 and C1, but not A6.

Having observed this Wigner remarked that from a physical point of view the Heisenberg equation and the Hamiltonian equations have a more direct physical significance than the CCR's. Therefore it is justified to postulate from the very beginning that the equations C1 hold instead of the CCRs. Based on all this we introduce

Definition 1. *A quantum system subject to axioms A1-A5 and C1 is said to be a Wigner Quantum System (WQS). A WQS is noncanonical if it does not satisfy A6, i.e. the canonical commutation relations.*

This definition was introduced for the first time in [4].

The statistics of canonical quantum mechanics is "hidden" in axiom A6, since the related creation and the annihilation operators (CAOs)

$$b_i^+ = \frac{1}{\sqrt{2}}(q_k - ip_k), \ \text{ and } \ b_i^- = \frac{1}{\sqrt{2}}(q_k + ip_k) \tag{4}$$

obey the Bose commutation relations:

$$[b_i^-, b_j^+] = \delta_{ij}, \ \ [b_i^-, b_j^-] = [b_i^+, b_j^+] = 0. \tag{5}$$

From A5 and C1 we conclude that a necessary condition a given quantum system to be a Wigner quantum system is *the main quantization condition* to hold, namely:

$$\frac{\mathrm{i}}{\hbar}[\hat{p}_k, H] = \frac{\partial H}{\partial \hat{q}_k} \tag{6a}$$

$$-\frac{\mathrm{i}}{\hbar}[\hat{q}_k, H] = \frac{\partial H}{\partial \hat{p}_k} \tag{6b}$$

These two equations actually correspond to the quantization in QM. Note that they depend on the Hamiltonian. In fact we know that these equations have a solution for Hamiltonians, corresponding to harmonic oscillators. What are the admissible Hamiltonians in the general case is an open question. This is hardly a surprise. In quantum field theory the harmonic oscillator interactions correspond to free fields interactions and so far one knows how to quantize rigorously only free fields.

2 B-Statistics

The next big step towards generalization of quantum statistics is due to Green [5], who in 1953 discovered the paraFermi (pF) and the paraBose (pB) statistics as new possible statistics in the free quantum field theory (QFT). In this more general setting the Fermi anticommutation relations were replaced by the more general double commutation relations for paraFermi CAO's, namely:

$$\begin{gathered}
[[f_i^+, f_j^-], f_k^+] = 2\delta_{jk} f_i^+, \\
[[f_i^+, f_j^-], f_k^-] = -2\delta_{ik} f_j^-, \\
[[f_i^+, f_j^+], f_k^+] = [[f_i^-, f_j^-], f_k^-] = 0.
\end{gathered} \tag{7}$$

Similarly the Bose commutation relations were replaced by similar relations, but part of the commutation relations have to be replaced by anticommutation relations:

$$\begin{gathered}
[\{b_i^+, b_j^-\}, b_k^+] = 2\delta_{jk} b_i^+, \\
[\{b_i^+, b_j^-\}, b_k^-] = -2\delta_{ik} b_j^-, \\
[\{b_i^+, b_j^+\}, b_k^+] = [\{b_i^-, b_j^-\}, b_k^-] = 0.
\end{gathered} \tag{8}$$

From the above triple relations there comes for the first time an indication about possible connection between quantum statistics and Lie (super)algebras [6].

Indeed, it is almost evident from (7) that the linear envelope

$$lin.env.\{f_i^\xi, [f_j^\eta, f_k^\epsilon]\; i,j,k = 1, ..., n, \xi, \eta, \epsilon = +, -\} \tag{9}$$

of all CAOs and their commutators is a Lie algebra(LA). It takes some time to prove that this LA is the algebra of the orthogonal group $so(2n+1)$ denoted also as B_n, if in (7) $i, j, k = 1, 2, \ldots, n$. For this reason it is natural to call the paraFemi statistics also $B-$statistics

The circumstance that the paraFermi operators have to satisfy the triple relations (7) means actually that these operators determine a representation of $so(2n+1)$. In the very definition of paraFermi statistics it is required that the representations are of Fock type [5]. This is achieved from the requirement that the state space contains a vacuum vector, so that

$$f_i^- f_j^+ |0\rangle = \delta_{ij} p |0\rangle, \tag{10}$$

where p is a positive integer. The number p is said to be *an order of statistics* [7]. It labels the different representations. The representations corresponding to different p are inequivalent. The pF statistics with an order of statistics $p = 1$ is the Fermi statistics.

As mentioned above the order of statistics is a positive integer. The representations with noninteger order of statistics do also exist and in certain cases could be of interest too [8]

The transformations of the Fock space under the action of the paraFermi CAO's is not simple and is in fact very difficult task. For more details we refer to [9]. The main difficulty stems from the observations that the paraFermi creation operators do not commute with each other. In view of this the very definition of the Fock space and its interpretation is nontrivial [5], [9].

Looking into some more mathematical books on Lie algebras, we learn that the algebra B_n belongs to the class B of the only four infinite classes of simple LA A, B, C and D.

Based on the above observations we draw the following

Conclusion 1. The paraFermi quantization corresponds to quantization with position and momentum operators (or to the related creation and annihilation operators) which generate a Lie algebra from the class B.

For this reason we give

Definition 2. We call the paraFermi statistics also a $B-$statistics and the related quantization a $B-$quantization .

The above conclusion rises an immediate

Question 1. Do there exist quantum statistics with position and momentum operators (or creation and annihilation operators) ,which generate Lie algebras from the classes A, C or D.

A positive answer to the above question was given in [1].

Definition 3. *The statistics, which position and momentum operators (or creation and annihilation operators) generate Lie algebras from the classes A, C or D are called $A-$, $C-$ or $D-$statistics, respectively.*

Let us now turn and analyze shortly the algebraic structure of paraBose (pB) operators. The defining relations for pF and pB operators (7) and (8) look pretty similar apart from the circumstance that part of the commutators in (7) are replaced by anticommutators in (8). This difference however turns to be essential. It is clear what we are searching for. We have to find all operators (8), i.e., we have to determine all representations of the relations (8) and then select those of them, which obey the restrictions of QM. How to determine the representations? The idea is to reduce the unknown problem to a more or less known one, similar as we did for paraFermi operators. To this end we first introduce appropriate terminology and notation.

To begin with we define two subspaces, called an even one B_0 and an odd one B_1, namely

$$B_0 = lin.env.\{\{b_i^{\pm}, b_j^{\pm}\}|\, i,j = 1,2,...,n,\}$$

$$B_1 = lin.env.\{b_k^{\pm}|k = 1,2,...,n.\} \tag{11}$$

Let $\mathbf{B}_n$ be their direct sum (in the sense of linear spaces)

$$\mathbf{B}_n = B_0 \oplus B_1. \tag{12}$$

The elements from B_0 (resp B_1) are also said to be *even (resp odd) elements*. Define a supercommutator $[\![x,y]\!]$ on $\mathbf{B_n}$, setting

$$[\![x,y]\!] = \{x,y\}, \text{ if } x,y \text{ are both odd}, \tag{13}$$

$$[\![x,y]\!] = [x,y], \text{ if } x \text{ or } y \text{ or both are even }. \tag{14}$$

and extend the above relations to every two elements by linearity. Then by definition $\mathbf{B}_n$ is a Lie superalgebra (LS) and the problem to determine all paraBose operators reduces to the task to determine the representations of the LS $\mathbf{B}_n$.

Here we had luck! Without going into the details, because we do not need them, we mention that $\mathbf{B}_n$ belongs to the class $\mathbf{B}$ of basic Lie superalgebras [10] and therefore the paraBose statistics can be called $\mathbf{B} - superstatistics$. There are four classes of basic Lie superalgebras in the classification of Kac [10], namely the classes $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$. Hence one can ask whether it is possible to associate creation and annihilation operators with each class of basic LS's and talk about $\mathbf{A}-$, $\mathbf{B}-$, $\mathbf{C}-$ and $\mathbf{D}-\mathbf{statistics}$, respectively

Finally, each Lie algebra is a Lie superalgebra with only even elements. Moreover it turns out that each class of simple Lie algebras is a subset of the corresponding class of basic Lie superalgebras, namely

$$A \subset \mathbf{A},\; B \subset \mathbf{B},\; C \subset \mathbf{C},\; D \subset \mathbf{D}. \tag{15}$$

Therefore, it is possible to unify the Lie algebra case with Lie superalgebra one, setting:

Question 2. Do there exist quantum statistics with position and momentum operators (or creation and annihilation operators) ,which generate Lie (super)algebras from the classes $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, or $\mathbf{D}$?

The answer to the above question is positive, but not complete. So far we have studied in more details the statistics, corresponding to the class A Lie algebras and Lie superalgebras. Let us say some more words about $\mathbf{A}$-(super)statistics.

3 A-Statistics

The CAO's of A-statistics satisfy the relations:

$$
\begin{aligned}
&[[a_i^+, a_j^-], a_k^+] = \delta_{jk} a_i^+ + \delta_{ij} a_k^+, \\
&[[a_i^+, a_j^-], a_k^-] = -\delta_{ik} a_j^- - \delta_{ij} a_k^-, \qquad (16)\\
&[a_i^+, a_j^+] = [a_i^-, a_j^-] = 0.
\end{aligned}
$$

Here we review shortly some of the properties of these CAO's and of their Fock representations.

The first remarkable property of A-statistics is that the creation operators commute with each other. This simplifies greatly all computations of A statistics and in particular the construction and the interpretation of the Foch space.

As indicated, the name A-statistics comes to remind that the operators $a_1^{\pm}, \ldots, a_n^{\pm}$ (and more generally any n different pairs of A-CAO's) generate a Lie algebra $A_n \equiv sl(n+1)$ from the class $\mathcal{A}$. In particular a set of $sl(n+1)$ generators, which constitute a linear basis in the underlying linear space, can be taken to be ($i \neq j = 1, \ldots, n$):

$$
e_{i0} = a_i^+, \; e_{0i} = a_i^-, \; e_{ii} - e_{00} = [a_i^+, a_i^-], \; e_{ij} = [a_i^+, a_j^-]. \qquad (17)
$$

Above $\{e_{ab} | a, b = 0, 1, \ldots, n\}$ are generators of $gl(n+1)$:

$$
[e_{ab}, e_{cd}] = \delta_{cb} e_{ad} - \delta_{ad} e_{cb}. \qquad (18)
$$

By definition each Fock space [11] is an irreducible $sl(n+1)-$module, which satisfies certain natural for physics requirements, namely the metric in each Fock space W is positive definite, W contains an unique vector $|0\rangle$, called a vacuum, so that $a_i^-|0\rangle = 0$ and the Hermitian conjugate to a_k^- is a_k^+ (for any integer k), i.e., $(a_k^+)^\dagger = a_k^-$. It is proved [11] that the irreducible inequivalent Fock spaces W_p are labeled by all positive integers $p = 1, 2, ...$, the order of statistics. Each Fock space W_p is a finite-dimensional irreducible state space. All vectors

$$
|p; l_1, .., l_n) = \sqrt{\frac{(p - \sum_{j=1}^n l_j)!}{p! l_1! l_2! \ldots l_n!}} (a_1^+)^{l_1} \ldots (a_n^+)^{l_n} |0\rangle, \qquad (19)
$$

subject to the condition $l_1 + \ldots + l_n \leq p$ constitute an orthonormal basis in W_p. The transformations of the basis (19) under the action of the CAOs read:

$$
a_i^+ |p; l_1, ., , l_i, .., l_n) = \sqrt{(l_i + 1)(p - \sum_{j=1}^n l_j)} \, |p; ., l_i + 1, .., l_n), \qquad (20a)
$$

$$a_i^-|p;l_1,..,l_i,..,l_n) = \sqrt{l_i(p - \sum_{j=1}^{n} l_j + 1)}\ |p;l_1,.,l_i-1,.,l_n). \qquad (20b)$$

In a consistent with (18) way we extend W_p to an irreducible $gl(n+1)$-module, setting for the central element

$$e_{00} + e_{11} + ... + e_{nn} = p. \qquad (21)$$

Then

$$e_{00}|p;l_1,l_2,...,l_n) = (p - \sum_{i=1}^{n} l_i)|p;l_1,l_2,...,l_n), \qquad (22)$$

$$e_{ii}|p;l_1,l_2,...,l_n) = l_i|p;l_1,l_2,...,l_n), \ \ i = 1,..,n. \qquad (23)$$

The operators $e_{00}, e_{11}, ..., e_{nn}$ commute. From (23) we conclude that e_{ii} can be interpreted as a number operator for the particles on the orbital i. Then $|p;l_1,\ldots l_{i-1},l_i,l_{i+1},\ldots,l_n\rangle$ is a state containing l_1 particles on orbital 1, l_2 particles on orbital 2, and so on,..,l_n particles on orbital n. Note that for a given p,

$$(a_1^+)^{l_1}\ldots(a_n^+)^{l_n}|0\rangle = 0, \ \ if \ \ l_1 + l_2 + ... + l_n > p. \qquad (24)$$

One immediate conclusion from (24) is evident:
Corrolary A. For any p the state space W_p is a finite-dimensional linear space.

The second conclusion from (24) is actually the

Pauli principle for $A-$statistics: If the order of statistics is p, then each basis state $|p;l_1,l_2,...,l_n)$ from W_p corresponds to $l_1 + l_2 + ... + l_n \leq p$ particles. There are no states with more than p particles in W_p. This issue holds certainly only in a particle interpretation of the picture.

Another relevant property of $A-$statistics is that in the limit $p \to \infty$ the operators

$$b(p)_k^{\pm} = a(p)_k^{\pm}/\sqrt{p}, \qquad (25)$$

are becoming ordinary bosons.

4 A-Superstatistics

Here we shall indicate how one can quantize a $3D$ harmonic oscillator based on $A-$superstatistics and more precisely with the CAO's of the LS $sl(1|3)$.

The creation and the annihilation operators of $A-$superstatistics, and more precisely of $sl(1|n)$ (with $i,j,k = 1,2,...,n$ bellow) read:

$$[\{a_i^+, a_j^-\}, a_k^+] = \delta_{jk}a_i^+ - \delta_{ij}a_k^+, \qquad (26a)$$

$$[\{a_i^+, a_j^-\}, a_k^-] = -\delta_{ik}a_j^- + \delta_{ij}a_k^-, \qquad (26b),$$

$$\{a_i^+, a_j^+\} = \{a_i^-, a_j^-\} = 0, \qquad (26c)$$

The linear span of all CAO's are the odd elements and all the anticommutators between the odd elements yields all even elements. In the case $i, j, k = 1, 2, 3$ the CAO's generate the LS $sl(1|3)$.

The Fock representation, which we consider, are determined from the requirements

$$a_i^-|0\rangle = 0, \;\; a_i^- a_j^+|0\rangle = \delta_{ij}p|0\rangle, \;\; i, j = 1, 2, 3. \qquad (27)$$

Moreover the hermiticity condition $(a_i^+)^* = a_i^-$ with $(*)$ being hermitian conjugation, should hold.

To each positive integer p, called *an order of statistics,* there corresponds an irreducible representation space $W(p)$. The representations corresponding to different p are inequivalent.

Let $\Theta \equiv \theta_1, \theta_2, \theta_3$ with $\theta_i \in (0, 1)$ for any $i = 1, 2, 3$. Define an orthonormed basis in $W(p)$:

$$|p, \Theta\rangle \equiv |p; \theta_1, \theta_2, \theta_3\rangle = \sqrt{\frac{(p-q)!}{p!}}(a_1^+)^{\theta_1}(a_2^+)^{\theta_2}(a_3^+)^{\theta_3}|0\rangle, \qquad (28)$$

where

$$0 \le q \equiv \theta_1 + \theta_2 + \theta_3 \le min(p, 3).$$

The transformation of the basis under the action of the CAO's reads:

$$a_i^-|p; ..., \theta_i ...\rangle = \theta_i(-1)^{\theta_1+...+\theta_i}\sqrt{p-q+1}|p; ..., \theta_i - 1, ...\rangle, \qquad (29a)$$

$$a_i^+|p; ..., \theta_i ...\rangle = (1-\theta_i)(-1)^{\theta_1+...+\theta_i}\sqrt{p-q}|p; ..., \theta_i + 1, ...\rangle. \qquad (29b)$$

We proceed to show that a (3D) harmonic oscillator can be quantized with the CAO's of $A-$ superstatistics defined above. This means that the physical observables related to the oscillator can be expressed via the the CAO's of $A-$superstatistics so that the oscillator will be a Wigner quantum system. We shall see that the properties of this oscillator will be very unusual. The Hamiltonian

$$H = \frac{\mathbf{P}^2}{2m} + \frac{m\omega^2}{2}\mathbf{R}^2, \qquad (30)$$

the equation of motion

$$\dot{\mathbf{P}} = -m\omega^2\mathbf{R}, \;\; \dot{\mathbf{R}} = \frac{1}{m}\mathbf{P} \qquad (31)$$

and the Heisenberg equations

$$\dot{\mathbf{P}} = \frac{i}{\hbar}[H, \mathbf{P}], \;\; \dot{\mathbf{R}} = \frac{i}{\hbar}[H, \mathbf{R}] \qquad (32)$$

are well known. For the compatibility equations we have

$$[H,\mathbf{P}] = i\hbar m\omega^2\mathbf{R},\ \ [H,\mathbf{R}] = -\frac{i\hbar}{m}\mathbf{P}. \tag{33}$$

In order to express the Hamiltonian, the position and the momentum operators via CAO' we set:

$$a_k^{\pm} = \sqrt{\frac{m\omega}{2\hbar}}R_k \pm i\sqrt{\frac{1}{2m\omega\hbar}}P_k,\ \ k = 1,2,3. \tag{34}$$

Then the Hamiltonian and the compatibility relations read:

$$H = \frac{\omega\hbar}{2}\sum_{i=1}^{3}\{a_i^+, a_i^-\} \tag{35}$$

$$\sum_{i=1}^{3}[\{a_i^+, a_i^-\}, a_k^{\pm}] = \mp 2a_k^+,\ \ i,k = 1,2,3. \tag{36}$$

Observe that in the derivation of the above relations we have not used the triple relations (26). Taking now into account the triple relations we immediately see that the compatibility equations are satisfied. The solution of the Heisenberg and the Hamiltonian equations are:

$$R(t)_k = \sqrt{\frac{\hbar}{2m\omega}}(a_k^+ e^{-i\omega t} + a_k^- e^{i\omega t}), \tag{37}$$

$$P(t)_k = -i\sqrt{\frac{\hbar m\omega}{2}}(a_k^+ e^{-i\omega t} + a_k^- e^{i\omega t}), \tag{38}$$

From all stated above we conclude that the oscillator has solutions as a Wigner quantum system. Let us mention some of its properties. On the first place, contrary to the canonical oscillator each irreducible state space $W(p)$ has no more than four equally spaced energy levels with spacing $\omega\hbar$, more precisely,

$$H|p;\theta_1,\theta_2,\theta_3\rangle = \frac{\omega\hbar}{2}(3p-2q)|p;\theta_1,\theta_2,\theta_3\rangle,\ \ q = \theta_1+\theta_2+\theta_3 = 0,1,...,min(p,3). \tag{39}$$

The next relation gives more information:

$$\frac{2}{\omega\hbar}H = \frac{2m\omega}{\hbar}\mathbf{R}^2 = \frac{2}{m\omega\hbar}\mathbf{P}^2 = \sum_{i=1}^{3}\{a_1^+, a_i^-\} \tag{40}$$

Therefore $H, \mathbf{R}^2$ and $\mathbf{P}^2$ commute with each other. We say that the geometry is square commutative, but not commutative. In particular

$$\mathbf{R}^2|p;\Theta) = \frac{\hbar}{2m\omega}(3p-2q)|p;\Theta). \tag{40}$$

$$\mathbf{P}^2|p;\Theta) = \frac{m\omega\hbar}{2}(3p-2q)$$

Therefore if $p > 2$, then $\mathbf{R}^2$ for instance can take no more then four different values. This means that the distance of the particle with respect to the center of the coordinate system is fixed and has no more than four values. One can be even more precise taking into account that all operators

$$H, \mathbf{R}^2, \mathbf{P}^2, R_1^2, R_2^2, R_3^2, P_1^2, P_2^2, P_3^2,$$

commute. If the oscillator is in a state $|p;\Theta>$ with $p > 2$, then

$$R_i^2|p;\theta_1,\theta_2,\theta_3>= \frac{\hbar}{2m\omega}(p-q+\theta_i), \quad i=1,2,3.$$

The conclusion is that if the particle is in a state $|p;\Theta>$, with $p > 2$, then at each measurement it can be spotted in no more than 8 different points of the (3D) space with coordinates along x, y, z axis in units $\sqrt{\frac{\hbar}{2m\omega}}$ as follows:

$$x = \pm\sqrt{p-q+\theta_1}, \quad y = \pm\sqrt{p-q+\theta_2}, \quad z = \pm\sqrt{p-q+\theta_3}.$$

5 Conclusion

We have indicated that from purely theoretical point of view there might exit new quantum statistics. The new statistics are selfconsistent, they have interesting predictions. The question however, formulated in the title remains open.

References

[1] T.D. Palev, Lie algebraical aspects of the quantum statistics. Habilitation Thesis (in Bulgarian): Inst. Nuclear Research and Nucl.Energy, Sofia, (1977).

[2] E.P. Wigner, *Phys. Rev.* **77** (1950) 711.

[3] P.A.M. Dirac, "*The Principles of Quantum Mechanics* (Oxford University Press, 1958).

[4] A.H. Kamupingene, T.D. Palev, S.P. Tsaneva, *J. Math. Phys.* **27** (1986) 2067.

[5] H.S. Green, *Phys. Rev.* **90** (1953) 270.

[6] S. Kamefuchi and Y. Takahashi, *Nucl. Phys.* **36** (1962) 177; C. Ryan and E.C.G. Sudarshan, *Nucl. Phys.* **47** (1963) 207.

[7] O.W. Greenberg and A.M.L. Messiah, *Phys. Rev. B* **138** (1965) 1155-1167.

[8] T.D. Palev, An example of an application of parastatistics of non-integer order (in Russian). High energies and Elementary particles p.331 (Proceedings of a Symposium, Varna, (1974).

[9] Y. Ohnuki, S. Kamefuchi, "*Quantum Field Theory and Parastatistics* (Springer, Berlin, 1982).

[10] V.G. Kac, *Adv. Math.* **26** (1977) 8-96.

[11] T.D. Palev, Lie algebraical aspects of quantum statistics. Unitary quantization (A-quantization). Preprint JINR E17-10550 (1977) and hep-th/9705032.