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Spectral Distribution Method for Neutrinoless Double Beta Decay: Results for ^{82}Se and ^{76}Ge

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Abstract. Statistical spectral distribution method based on shell model and random matrix theory is developed for calculating neutrinoless double beta decay nuclear transition matrix elements. First results obtained for ^{82}Se and ^{76}Ge using the spectral method are close to the available shell model results.

1 Introduction

Neutrinoless double beta decay ($0\nu\beta\beta$ or NDBD) which involves emission of two electrons without the accompanying neutrinos and which violates lepton number conservation has been an important and challenging problem both for the experimentalists and theoreticians. Recent neutrino oscillation experiments have demonstrated that neutrinos have mass [1–3]. The observation of $0\nu\beta\beta$ decay is expected to provide information regarding the absolute neutrino mass which is, as yet, not known. As a result, experimental programs to observe this decay have been initiated at different laboratories across the globe and already are in advanced stages of development. The most recent results for $0\nu\beta\beta$ decay of ^{136}Xe have been reported by EXO-200 collaboration [4] and KamLand-Zen collaboration [5]. They give a lower limit of 3.4×10^{25} yr for the half-life. Further, phase I results from GERDA experiment [6] for ^{76}Ge gives a lower limit of 3.0×10^{25} yr for the half-life. Nuclear transition matrix elements (NTME) are the essential ingredient for extracting the neutrino mass from the half lives [7]. There has been considerable effort to obtain NTME for various candidate nuclei and they have been calculated theoretically using a variety of nuclear models: (i) large scale shell model; (ii) quasi-particle random phase approximation and its variants; (iii) proton-neutron interacting boson model; (iv) particle number and angular momentum projection including configuration mixing within the generating coordinate method framework; (v) projected Hartree-Fock-Bogoliubov method with pairing plus quadrupole-quadrupole interaction. A detailed comparative study of the results from these various methods is discussed in [8, 9]. In addition, more recently the so called deformed shell model based on Hartree-Fock single particle states has been used for the candidate nuclei in the A=60-90 region [10]. It is important to note that the predictions of various models for NTME vary typically from 2 to 6 [8].

The statistical spectral distribution method (SDM) developed by French and collaborators for nuclear structure is well documented [11] and the operation of embedded Gaussian orthogonal ensemble of random matrices (EGOE) in nuclear shell model spaces forms the basis for SDM [12]. With this, it is natural that one should develop and apply SDM for calculating the NTME for $0\nu\beta\beta$ and compare the results with those obtained using shell model and other models. This is addressed in the present paper with first results for ^{82}Se and ^{76}Ge. The essential point is that NTME can be viewed as a transition strength (square of the matrix element connecting a given initial state to a final state by a transition operator) generated by the NDBD operator that is two-body in nature. Therefore, SDM for transition strengths as given in [11, 13–15] can be used as the starting point for further developments and applications. Let us add that SDM is sometimes called moment method.

We now give a preview. Section 2 gives a brief discussion of the relation between neutrino mass and NTME and then deals with the structure of the NDBD transition operator. Section 3 deals with the details of SDM for transition strengths as applicable for NDBD. In Section 4, we present SDM results for ^{82}Se and ^{76}Ge NDBD NTME. For these two nuclei, experiments SuperNEMO and GERDA+MAJORANA respectively are under development to measure NDBD half lives. Finally, conclusions and future outlook are given in Section 5.

2 Neutrinoless Double Beta Decay and NTME

In $0\nu\beta\beta$, the half-life for the 0_i^+ ground state (gs) of a initial even-even nucleus decay to the 0_f^+ gs of the final even-even nucleus is given by [7]

$$\left[T^{0\nu}_{1/2}(0_i^+ \to 0_f^+)\right]^{-1} = G^{0\nu} \left|M^{0\nu}(0^+)\right|^2 \left(\frac{\langle m_\nu \rangle}{m_e}\right)^2 , \tag{1}$$

where $\langle m_\nu \rangle$ is the effective neutrino mass (a combination of neutrino mass eigenvalues and also involving the neutrino mixing matrix). The $G^{0\nu}$ is a phase space integral (kinematical factor); tabulations for $G^{0\nu}$ are available. The $M^{0\nu}$ represents NTME of the NDBD transition operator and it is a sum of a Gamow-Teller like (M_{GT}), Fermi like (M_F) and tensor (M_T) two-body operators. Since it is well known that the tensor part contributes only up to 10% of the matrix elements, we will neglect the tensor part. Then, from the closure approximation which is well justified for NDBD, we have

$$\begin{aligned} M^{0\nu}(0^+) &= M^{0\nu}_{GT}(0^+) - \frac{g_V^2}{g_A^2} M^{0\nu}_F(0^+) = \left\langle 0_f^+ \,||\, \mathcal{O}(2:0\nu) \,||\, 0_i^+ \right\rangle , \\ \mathcal{O}(2:0\nu) &= \sum_{a,b} \mathcal{H}(r_{ab}, \overline{E}) \tau_a^+ \tau_b^+ \left(\sigma_a \cdot \sigma_b - \frac{g_V^2}{g_A^2}\right) . \end{aligned} \tag{2}$$

As seen from Eq. (2), NDBD half-lives are generated by the two-body transition operator $\mathcal{O}(2:0\nu)$; note that a, b label nucleons. The g_A and g_V are the weak

axial-vector and vector coupling constants. The $\mathcal{H}(r_{ab}, \overline{E})$ in Eq. (2) is called the 'neutrino potential'. Here, $\overline{E}$ is the average energy of the virtual intermediate states used in the closure approximation. The form given by Eq. (2) is justified *only if the exchange of the light majorana neutrino is indeed the mechanism responsible for the NDBD*. With the phase space factors fairly well known, all one needs are NTME $|M^{0\nu}(0^+)| = \left|\left\langle 0_f^+ \,||\, \mathcal{O}(2:0\nu) \,||\, 0_i^+ \right\rangle\right|$. Then, measuring the half-lives makes it possible to deduce neutrino mass using Eq. (1).

The neutrino potential is of the form $\mathcal{H}(r_{ab}, \overline{E}) = [R/r_{ab}]\, \Phi(r_{ab}, \overline{E})$ where R in fm units is the nuclear radius and similarly r_{ab} is in fm units. A simpler form for the function Φ, involving sine and cosine integrals, as given in [7] and employed in [10], is used in the present work. It is useful to note that $\Phi(r_{ab}, \overline{E}) \sim \exp(-\frac{3}{2}\frac{\overline{E}}{\hbar c} r_{ab})$. The effects of short-range correlations in the wavefunctions are usually taken into account by multiplying the wavefunction by the Jastrow function $[1 - \gamma_3 e^{-\gamma_1 r_{ab}^2}(1 - \gamma_2 r_{ab}^2)]$. There are other approaches [16] for taking into account the short range correlations but they are not considered here. Now, keeping the wavefunctions unaltered, the Jastrow function can be incorporated into $\mathcal{H}(r_{ab}, \overline{E})$ giving an effective $\mathcal{H}_{eff}(r_{ab}, \overline{E})$,

$$\mathcal{H}(r_{ab}, \overline{E}) \rightarrow \mathcal{H}_{eff}(r_{ab}, \overline{E}) = \mathcal{H}(r_{ab}, \overline{E})[1 - \gamma_3\, e^{-\gamma_1\, r_{ab}^2}(1 - \gamma_2\, r_{ab}^2)]^2 \ . \quad (3)$$

The choice of the values for the parameters γ_1, γ_2 and γ_3 is given in Section 4.

Let us say that for the nuclei under consideration, protons are in the single particle (sp) orbits j^p and similarly neutrons in j^n. Using the usual assumption that the radial part of the sp states are those of the harmonic oscillator, the proton sp states are completely specified by $(\mathbf{n}^p, \ell^p, j^p)$ with $\mathbf{n}^p$ denoting oscillator radial quantum number so that for a oscillator shell $\mathcal{N}^p$, $2\mathbf{n}^p + \ell^p = \mathcal{N}^p$. Similarly, the neutron sp states are $(\mathbf{n}^n, \ell^n, j^n)$. In terms of the creation $(a^\dagger)$ and annihilation (a) operators, normalized two-particle (antisymmetrized) creation operator $A_\mu^J(j_1 j_2) = (1 + \delta_{j_1 j_2})^{-1/2}(a_{j_1}^\dagger a_{j_2}^\dagger)_\mu^J$ and then $A_\mu^J\,|0\rangle = |(j_1 j_2) J\mu\rangle$ represents a normalized two-particle state. At this stage, it is important to emphasize that we are considering only 0^+ to 0^+ transitions in $0\nu\beta\beta$ and therefore only the J scalar part of $\mathcal{O}(2:0\nu)$ will contribute to $M^{0\nu}$. With this, the NDBD transition operator can be written as,

$$\mathcal{O}(2:0\nu) = \sum_{j_1^p \geq j_2^p; j_3^n \geq j_4^n; J} \mathcal{O}^J_{j_1^p\, j_2^p; j_3^n\, j_4^n}(0\nu) \sum_\mu A_\mu^J(j_1^p j_2^p)\left\{A_\mu^J(j_3^n j_4^n)\right\}^\dagger \ . \quad (4)$$

Here, $\mathcal{O}^J_{j_1^p\, j_2^p; j_3^n\, j_4^n}(0\nu) = \langle (j_1^p\, j_2^p) JM \mid \mathcal{O}(2:0\nu) \mid (j_3^n\, j_4^n) JM \rangle_a$ are two-body matrix elements (TBME) and 'a'denotes antisymmetrized two-particle wavefunctions; J is even for $j_1 = j_2$ or $j_3 = j_4$. The TBME are obtained by using the standard approach based on Brody-Moshinisky brackets and Talmi integrals.

3 Spectral Distribution Method for NDBD

3.1 State densities and Gaussian form

Let us consider shell model sp orbits $j_1^p, j_2^p, \ldots, j_r^p$ with m_p protons distributed in them. Similarly, m_n neutrons are distributed in $j_1^n, j_2^n, \ldots, j_s^n$ orbits. Then, the proton configurations are $\widetilde{m_p} = [m_p^1, m_p^2, \ldots, m_p^r]$ where m_p^i is number of protons in the orbit j_i^p with $\sum_{i=1}^r m_p^i = m_p$. Similarly, the neutron configurations are $\widetilde{m_n} = [m_n^1, m_n^2, \ldots, m_n^s]$ where m_n^i is number of neutrons in the orbit j_i^n with $\sum_{i=1}^s m_n^i = m_n$. With these, $(\widetilde{m_p}, \widetilde{m_n})$'s denote proton-neutron configurations. The nuclear effective Hamiltonian is one plus two-body, $H = h(1)+V(2)$ and we assume that the one-body part $h(1)$ includes the mean-field producing part of the two-body interaction. Thus, $V(2)$ is the irreducible two-body part of H [11]. From now on, for simplicity we shall denote $h = h(1)$ and $V = V(2)$. The state density $I^H(E)$, with $\langle\langle - - \rangle\rangle$ denoting trace, can be written as a sum of the partial densities defined over $(\widetilde{m_p}, \widetilde{m_n})$,

$$I^{(m_p,m_n)}(E) = \langle\langle \delta(H-E) \rangle\rangle^{(m_p,m_n)} = \sum_{(\widetilde{m_p},\widetilde{m_n})} \langle\langle \delta(H-E) \rangle\rangle^{(\widetilde{m_p},\widetilde{m_n})}$$
$$= \sum_{(\widetilde{m_p},\widetilde{m_n})} I^{(\widetilde{m_p},\widetilde{m_n})}(E) = \sum_{(\widetilde{m_p},\widetilde{m_n})} d(\widetilde{m_p},\widetilde{m_n})\, \rho^{(\widetilde{m_p},\widetilde{m_n})}(E)\ . \quad (5)$$

Here, $d(\widetilde{m_p}, \widetilde{m_n})$ is the dimension of the configuration $(\widetilde{m_p}, \widetilde{m_n})$ and $\rho^{(\widetilde{m_p},\widetilde{m_n})}(E)$ is normalized to unity. For strong enough two-body interactions (this is valid for nuclear interactions [11]), the operation of embedded GOE of one plus two-body interactions [EGOE(1+2)] will lead to Gaussian form for the partial densities $\rho^{(\widetilde{m_p},\widetilde{m_n})}(E)$ and therefore,

$$I^{(m_p,m_n)}(E) = \sum_{(\widetilde{m_p},\widetilde{m_n})} I_{\mathcal{G}}^{(\widetilde{m_p},\widetilde{m_n})}(E)\ . \quad (6)$$

In Eq. (6), $\mathcal{G}$ denotes Gaussian. The Gaussian partial densities are defined by the centroids $E_c(\widetilde{m_p}, \widetilde{m_n}) = \langle H \rangle^{(\widetilde{m_p},\widetilde{m_n})}$ and variances $\sigma^2(\widetilde{m_p}, \widetilde{m_n}) = \langle H^2 \rangle^{(\widetilde{m_p},\widetilde{m_n})} - [E_c(\widetilde{m_p}, \widetilde{m_n})]^2$. Expressions for these follow easily from trace propagation methods [11, 17]. In practical applications to nuclei, Eq. (6) has to be applied in fixed-J spaces [18, 19] or an approximate J projection has to be carried out [11, 20, 21]. We will return to this question in Section 3.4.

3.2 Transition strength densities and bivariate Gaussian form

Given a transition operator $\mathcal{O}$, the spectral distribution method for transition strengths starts with the transition strength density $I_{\mathcal{O}}^H(E_i, E_f)$,

$$I_{\mathcal{O}}^H(E_i, E_f) = I(E_f) |\langle E_f \mid \mathcal{O} \mid E_i \rangle|^2 I(E_i) \quad (7)$$

where E's are eigenvalues of H. A plausible way to proceed now [13] is to first construct the transition strength density with $H = h = \sum_r \epsilon_r n_r$; n_r is the number operator for the orbit r and ϵ_r are the sp energies (spe). As the configurations $(\widetilde{m_p}, \widetilde{m_n})$ are eigenstates of h, it is straightforward to construct $I^h_{\mathcal{O}}$ [11]. Next the interaction V, the two-body part of H, is switched on. Then, the role of V is to locally spread $I^h_{\mathcal{O}}$ and therefore the strength density will be a bivariate convolution of $I^h_{\mathcal{O}}$ and $\rho^V_{\mathcal{O}}$; the spreading function (normalized to unity) $\rho^V_{\mathcal{O}}$ is a bivariate distribution. For strong enough interactions, operation of EGOE(1+2) generates bivariate Gaussian form for $\rho^V_{\mathcal{O}}$ and this result has been established for NDBD type operators in [15]. Applying this, with some additional approximations as discussed ahead, will give

$$|\langle E_f \mid \mathcal{O} \mid E_i\rangle|^2 = \sum_{\widetilde{m}_i,\widetilde{m}_f} \frac{I^{\widetilde{m}_i}_{\mathcal{G}}(E_i) I^{\widetilde{m}_f}_{\mathcal{G}}(E_f)}{I^{m_i}(E_i) I^{m_f}(E_f)} \, |\langle \widetilde{m}_f \mid \mathcal{O} \mid \widetilde{m}_i\rangle|^2$$

$$\times \frac{\rho^V_{\mathcal{O}:biv-\mathcal{G}}(E_i, E_f, \mathcal{E}_{\mathcal{O}:V}(\widetilde{m}_i), \mathcal{E}_{\mathcal{O}:V}(\widetilde{m}_f), \sigma_{\mathcal{O}:V}(\widetilde{m}_i), \sigma_{\mathcal{O}:V}(\widetilde{m}_f), \zeta_{\mathcal{O}:V}(\widetilde{m}_i, \widetilde{m}_f))}{\rho^{\widetilde{m}_i}_{\mathcal{G}}(E_i) \rho^{\widetilde{m}_f}_{\mathcal{G}}(E_f)} \; ;$$

$$|\langle \widetilde{m}_f \mid \mathcal{O} \mid \widetilde{m}_i\rangle|^2 = [d(\widetilde{m}_i) d(\widetilde{m}_f)]^{-1} \sum_{\alpha,\beta} |\langle \widetilde{m}_f, \alpha \mid \mathcal{O} \mid \widetilde{m}_i, \beta\rangle|^2 \; . \quad (8)$$

This is the basic equation that allows one to use SDM for the calculation of NTME $M^{0\nu}$. In order to apply this, we need the marginal centroids $\mathcal{E}_{\mathcal{O}:V}(\widetilde{m}_i)$ and $\mathcal{E}_{\mathcal{O}:V}(\widetilde{m}_f)$, marginal variances $\sigma^2_{\mathcal{O}:V}(\widetilde{m}_i)$ and $\sigma^2_{\mathcal{O}:V}(\widetilde{m}_f)$ and the correlation coefficient $\zeta_{\mathcal{O}:V}(\widetilde{m}_i, \widetilde{m}_f)$ defining $\rho^V_{\mathcal{O}:biv-\mathcal{G}}$. Also, we need $|\langle \widetilde{m}_f \mid \mathcal{O} \mid \widetilde{m}_i\rangle|^2$. Note that $\widetilde{m} = (\widetilde{m_p}, \widetilde{m_n})$ in actual applications and further, the angular momentum quantum numbers for the parent and daughter nuclei involved in $0\nu\beta\beta$ decay need to be considered. We will turn to these now.

3.3 SDM for NTME for $0\nu\beta\beta$ decay

Firstly, the marginal centroids and variances in Eq. (8) are approximated, following random matrix theory [15], to the corresponding state density centroids and variances (see for example [13,14]) giving $\mathcal{E}_{\mathcal{O}:V}((\widetilde{m_p}, \widetilde{m_n})_r) \approx E_c((\widetilde{m_p}, \widetilde{m_n})_r)$ $= \langle H\rangle^{(\widetilde{m_p},\widetilde{m_n})_r}$ and $\sigma^2_{\mathcal{O}:V}((\widetilde{m_p}, \widetilde{m_n})_r)) \approx \sigma^2((\widetilde{m_p}, \widetilde{m_n})_r)) = \left\langle V^2\right\rangle^{(\widetilde{m_p},\widetilde{m_n})_r}$; $r = i, f$. For the correlation coefficient ζ, there is not yet any valid form involving configurations. Therefore, the only plausible way forward currently is to estimate ζ as a function of (m_p, m_n) using random matrix theory given in [13,15]. Then, the definition of ζ is

$$\zeta_{\mathcal{O}:V}(m_p, m_n) = \frac{\left\langle \mathcal{O}(2:0\nu)^\dagger \, V \, \mathcal{O}(2:0\nu) \, V \right\rangle^{(m_p,m_n)}}{\sqrt{\left\langle \mathcal{O}(2:0\nu)^\dagger \, V^2 \, \mathcal{O}(2:0\nu)\right\rangle^{(m_p,m_n)} \; \left\langle \mathcal{O}(2:0\nu)^\dagger \, \mathcal{O}(2:0\nu) \, V^2\right\rangle^{(m_p,m_n)}}} \; . \quad (9)$$

Results in Sections 5 and 6 of [15], obtained using EGOE representation for both the $\mathcal{O}$ and V operators, will allow one to obtain $\zeta(m_p, m_n)$. For nuclei of interest, using the numerical results in Table 2 of [15], it is seen that $\zeta \sim 0.6 - 0.8$. These values are used in the $M^{0\nu}$ calculations reported in Section 4 ahead.

In order to apply Eq. (8), in addition to the marginal centroids, variances and ζ, we also need an expression for $|\langle \widetilde{m}_f \mid \mathcal{O} \mid \widetilde{m}_i \rangle|^2$, the configuration mean square matrix element of the transition operator. Applying the propagation theory given in [17] will give,

$$
\begin{aligned}
&|\langle (\widetilde{m_p}, \widetilde{m_n})_f \mid \mathcal{O}(2:0\nu) \mid (\widetilde{m_p}, \widetilde{m_n})_i \rangle|^2 = \{ d[(\widetilde{m_p}, \widetilde{m_n})_f] \}^{-1} \\
&\times \sum_{\alpha,\beta,\gamma,\delta} \frac{m_n^i(\alpha)[m_n^i(\beta) - \delta_{\alpha\beta}][N_p(\gamma) - m_p^i(\gamma)][N_p(\delta) - m_p^i(\delta) - \delta_{\gamma\delta}]}{N_n(\alpha)[N_n(\beta) - \delta_{\alpha\beta}]\, N_p(\gamma)[N_p(\delta) - \delta_{\gamma\delta}]} \\
&\qquad\qquad \times \sum_{J_0} \left[\mathcal{O}^{J_0}_{\gamma^p \delta^p \alpha^n \beta^n}(0\nu) \right]^2 (2J_0 + 1) \; ;
\end{aligned}
$$

$$
(\widetilde{m_p}, \widetilde{m_n})_f = (\widetilde{m_p}, \widetilde{m_n})_i \times \left(1^+_{\gamma_p} 1^+_{\delta_p} 1_{\alpha_n} 1_{\beta_n} \right) \; . \tag{10}
$$

Note that in Eq. (10), the final configuration is defined by removing one neutron from orbit α and another from β and then adding one proton in orbit γ and another in orbit δ. Also, $N_p(\alpha)$ is the degeneracy of the proton orbit α and similarly $N_n(\gamma)$ for the neutron orbit γ.

3.4 Angular momentum decomposition of transition strengths

For NDBD NTME calculations, to complete the transition strength theory given by Eqs. (8) - (10), we need J projection as the quantity of interest is

$$
|\langle E_f J_f = 0 \mid \mathcal{O} \mid E_i J_i = 0 \rangle|^2
$$

where E_i and E_f are the ground state energies of the parent and daughter nuclei respectively and similarly J_i and J_f. Firstly, note that

$$
\begin{aligned}
&|\langle E_f J_f = 0 \mid \mathcal{O}(2:0\nu) \mid E_i J_i = 0 \rangle|^2 \\
&= \frac{\left\langle\!\left\langle [\mathcal{O}(2:0\nu)]^\dagger X(H, J^2, E_f, J_f) \mathcal{O}(2:0\nu) Y(H, J^2, E_i, J_i) \right\rangle\!\right\rangle^{(m_p^i m_n^i)}}{\langle\!\langle X(H, J^2, E_f, J_f) \rangle\!\rangle^{(m_p^f m_n^f)} \, \langle\!\langle Y(H, J^2, E_i, J_i) \rangle\!\rangle^{(m_p^i m_n^i)}} \\
&\qquad\qquad = \frac{I^{(m_p^i m_n^i),(m_p^f m_n^f)}_{\mathcal{O}(2:0\nu)}(E_i, E_f, J_i, J_f)}{I^{(m_p^i m_n^i)}(E_i, J_i) I^{(m_p^f m_n^f)}(E_f, J_f)} \; ;
\end{aligned}
$$

$$
\begin{aligned}
&X(H, J^2, E_f, J_f) = \delta(H - E_f)\delta(J^2 - J_f(J_f + 1)) \; , \\
&Y(H, J^2, E_i, J_i) = \delta(H - E_i)\delta(J^2 - J_i(J_i + 1)) \; .
\end{aligned} \tag{11}
$$

The four variate density $I_{\mathcal{O}}(E_i, E_f, J_i, J_f) = I_{\mathcal{O}}(E_i, E_f)\rho_{\mathcal{O}}(J_i, J_f : E_i, E_f)$ where ρ is a conditional density. Now, using the fact that J_f (J_i) is uniquely determined by J_i (J_f) for the $\mathcal{O}(2 : 0\nu)$ operator and the J-factoring used in [13] will give the approximation

$$I_{\mathcal{O}(2:0\nu)}(E_i, E_f, J_i, J_f) \sim I_{\mathcal{O}(2:0\nu)}(E_i, E_f)\sqrt{C_{J_i}(E_i)C_{J_f}(E_f)}\,. \qquad (12)$$

The function $C_J(E)$ involves spin cut-off factor as given below. In addition, we have the well established result [11, 13, 17, 21] $I(E, J) = I(E)C_J(E)$. Using these will give,

$$|\langle E_f J_f = 0 \mid \mathcal{O}(2 : 0\nu) \mid E_i J_i = 0\rangle|^2 = \frac{|\langle E_f \mid \mathcal{O}(2 : 0\nu) \mid E_i\rangle|^2}{\sqrt{C_{J_i=0}(E_i)C_{J_f=0}(E_f)}}\,;$$

$$C_{J_r}(E_r) = \frac{(2J_r+1)}{\sqrt{8\pi}\ \sigma_J^3(E_r)} \exp -(2J_r+1)^2/8\sigma_J^2(E_r)$$
$$\xrightarrow{J_r=0} \frac{1}{\sqrt{8\pi}\sigma_J^3(E_r)} \qquad (13)$$

where $r = i, f$. Note that $\sigma_J^2(E) = \left\langle J_Z^2 \right\rangle^E$ is the energy dependent spin cut-off factor. In the approximation $C_{J_i=0}(E_i) \sim [\sqrt{8\pi}\sigma_J^3(E_i)]^{-1}$ (similarly for $C_{J_f=0}(E_f)$), we have used the fact that in general $\sigma_J(E) >> 1$. The spin cut-off factor can be calculated using SDM [11, 20, 21]. Carrying this out for the nuclei of interest in the present study, it is seen that $\sigma_J(E) \sim 3 - 4$ with E varying up to 5 MeV excitation. Similarly, for lower $2p - 1f$ shell nuclei studied in [20], $\sigma_J(E) \sim 4 - 6$. Because of the uncertainties in using spin decomposition via Eq. (13), in the present work $M^{0\nu}$ is calculated by varying σ_J from 3 to 6. In principle it is possible to avoid the use of spin cut-off factors (see Section 5).

4 SDM Results for ^{82}Se and ^{76}Ge $0\nu\beta\beta$ NTME

In the first application of SDM given in Section 3, we have chosen ^{82}Se as large shell model results, obtained using an easily available and well established effective interaction, for the NTME for the $0\nu\beta\beta$ decay to ^{82}Kr are available in [22]. In addition, SuperNEMO experiment will be measuring ^{82}Se $0\nu\beta\beta$ decay half-life [23]. In the shell model calculations, ^{56}Ni is the core and the valence protons and neutrons in ^{82}Se and ^{82}Kr occupy the $f_{5/2}pg_{9/2}$ orbits $^1p_{3/2}$, $^0f_{5/2}$, $^1p_{1/2}$ and $^0g_{9/2}$. The effective interaction used is JUN45. The spe and TBME defining JUN45 are given in [24]. In the SDM application, same shell model space, spe and TBME are employed. Firstly, all the proton-neutron configurations are generated for both ^{82}Se and ^{82}Kr. Number of positive parity configurations is 316 for ^{82}Se and 1354 for ^{82}Kr.

Using the formula in [17] and the JUN45 interaction, the centroids and variances defining the Gaussian partial densities in Eq. (6) are calculated. These will also give the marginal centroids and variances in Eq. (8). The average width ($\overline{\sigma}$) for ^{82}Se configurations is 3.34 MeV with a 9% fluctuation. Similarly, for ^{82}Kr, $\overline{\sigma} = 4.7$ MeV with a 5% fluctuation. Proceeding further, the TBME $\mathcal{O}^{J}_{j_1^p\, j_2^p; j_3^n\, j_4^n}(0\nu)$ defining the $0\nu\beta\beta$ transition operator are calculated and they are 259 in number for the chosen set of sp orbits. The choices made for the various parameters in the transition operator are (i) $R = 1.2A^{1/3}$ fm; (ii) $b = 1.003A^{1/6}$ fm; (iii) $\overline{E} = 1.12A^{1/2}$ MeV; (iv) $g_A/g_V = 1$ (quenched); (v) $\gamma_1 = 1.1\, fm^{-2}$, $\gamma_2 = 0.68\, fm^{-2}$ and $\gamma_3 = 1$ (these are Miller-Spencer Jastrow correlation parameters). Then, applying Eq. (10), the configuration mean square matrix elements of the transition operator are obtained for all the configurations. With all these, in order to apply Eqs. (8) and (13), we need the ground states of ^{82}Se and ^{82}Kr and also the values of the ζ and σ_J parameters.

Using the so called Ratcliff procedure [11, 25], the ground states are determined in SDM. For this one needs a reference level with energy (E_R) and angular momentum and parity J^π value (J^π_R) and also the total number of states up to and including the reference level (N_R). The constraint in choosing the reference level is that the J^π values for all levels up to the reference level should be known with certainty. Satisfying this, we have, from the most recent data [26], for ^{82}Se the values $E_R = 1.735$MeV, $J^\pi_R = 4^+$ and $N_R = 21$. Similarly, for ^{82}Kr we have $E_R = 2.172$MeV, $J^\pi_R = 0^+$ and $N_R = 34$. The ground states are found to be $\sim 3\sigma$ below the lowest configuration centroid.

After obtaining the ground states, the ground to ground NTME are calculated using Eq. (8) with J-decomposition via Eq. (13). For the correlation coefficient ζ the values 0.6, 0.65, 0.7 and 0.8 are used as stated in Section 3.3. Similarly, assuming $\sigma_J(E_i(gs)) = \sigma_J(E_f(gs)) = \sigma_J$, the values chosen for σ_J are 3, 4, 5 and 6 as stated in Section 3.4. With increasing ζ and σ_J values, it is easy to see that the NTME $M^{0\nu}$ will increase. The values of NTME for $\zeta = 0.6$ and $\sigma_J = 3$, 4, 5 and 6 are 1, 1.54, 2.15 and 2.83 respectively. Similarly, for $\zeta = 0.65$, 0.7 and 0.8 they are $(1.18, 1.82, 2.54, 3.34)$, $(1.38, 2.12, 2.97, 3.9)$ and $(1.78, 2.74, 3.82, 5.03)$ respectively. With these and using $\sigma_J \sim 3 - 4$ and $\zeta \sim 0.7 - 0.8$ will give $M^{0\nu} \sim 2 - 3$ in SDM while the shell model value given in [22] by Horoi et al., using JUN45 interaction and same Jastrow parameters, is 2.59. It is important to note that the shell model results include a more detailed transition operator and other modifications. In addition, with a different interaction Poves et al. [27] obtained the shell model value to be ~ 2.18. As already stated in the introduction, with other nuclear models $M^{0\nu} \sim 3 - 6$. Thus, it is plausible to conclude that SDM is useful for calculating NTME for $0\nu\beta\beta$. For further confirmation of this, in a second example ^{76}Ge is considered and GERDA+MAJORANA experiments will measure the ^{76}Ge $0\nu\beta\beta$ decay half life in future [28].

For ^{76}Ge to ^{76}Se NDBD NTME, same shell model inputs are used as above and similarly the parameters in the transition operator. Number of positive par-

ity proton-neutron configurations is 958 for ^{76}Ge and 2604 for ^{76}Se. The $\overline{\sigma}$ for ^{76}Ge configurations is 4.4 MeV with a 6% fluctuation. Similarly, for ^{76}Se, $\overline{\sigma} = 5.51$ MeV with a 4% fluctuation. For the ground state determination we have [26], $(E_R, J^\pi_R, N_R) = (2.02\text{MeV}, 4^+, 37)$ for ^{76}Ge and $(E_R, J^\pi_R, N_R) = (1.79\text{MeV}, 2^+, 33)$ for ^{76}Se. The ground states here are also $\sim 3\sigma$ below the lowest configuration centroid. With all these, the NTME are calculated and their values for $\zeta = 0.65$ and $\sigma_J = 3$, 4 , 5 and 6 are 1.02, 1.56, 2.19 and 2.87 respectively. Similarly, for $\zeta = 0.7$ and 0.8 they are $(1.29, 1.98, 2.77, 3.63)$ and $(1.96, 3.01, 4.21, 5.54)$ respectively. Shell model result from Horoi et al. [29], obtained using JUN45 interaction and same Jastrow parameters, is 2.72 while it is 2.3 from Poves et al. [27] shell model calculations. Clearly, the SDM values with $\zeta \sim 0.7 - 0.8$ and $\sigma_j \sim 4$ are close to the shell model results.

5 Conclusions and Future Outlook

In the present paper SDM for calculating NTME for NDBD is described with all the relevant equations. As first examples, results for ^{82}Se and ^{76}Ge are presented and the SDM results are seen to be close to the shell model values.

It is clearly important that the SDM formulation given in Section 3 should be tested. This is possible by constructing complete shell model Hamiltonian matrix, in the configuration-J basis, for the parent and daughter nuclei (with J^π values fixed) and the transition matrix generated by the action of the transition operator on each of the parent states taking to the daughter states. Although this might seem complicated for realistic nuclei, a pseudo NDBD nucleus such as ^{24}Mg could used for the test.

Another direction for a better SDM calculation is to evaluate all the configuration centroids and variances with fixed-J using for example, the large scale computer codes developed recently by Sen'kov et al. [18]; note that we need $E_c((\widetilde{m}_p, \widetilde{m}_n), J = 0)$ and $\sigma((\widetilde{m}_p, \widetilde{m}_n), J = 0)$. However, the methods used by Sen'kov et al. need to be extended to derive a formula (or a viable method for computing) $|\langle(\widetilde{m}_p, \widetilde{m}_n)_f J_f = 0 \mid \mathcal{O} \mid (\widetilde{m}_p, \widetilde{m}_n)_i J_i = 0\rangle|^2$. With these, it is possible in the near future to apply the theory described in Section 3 without using Eq. (13) for the calculation of NTME.

Most important is to improve SDM theory with a better treatment of ζ including its definition with configuration partitioning, although ζ via Eq. (9) and its extensions could be made tractable. In future, this need to be addressed.

It is useful to add that Eq. (10) easily gives the total transition strength sum, the sum of the strengths from all states of the parent nucleus to all the states of the daughter nucleus and this depends only on the sp space considered. For the ^{82}Se the total strength sum is 31239 and for ^{76}Ge it is 54178. Thus, $M^{0\nu}(0^+)$ is a very small fraction of the total strength generated by the NDBD transition operator. Starting from Eq. (8), it is possible to obtain the total strength (NEWSR) originating from the ground state of the parent nucleus and also the linear and quadratic energy weighted strength sums. These may prove to be use-

ful in putting constraints on the nuclear models being used for NDBD studies. This will be addressed in future.

Finally, using SDM [11, 30] it is possible to study orbit occupancies and GT distributions in various NDBD nuclei. These results can be compared with available experimental data and will provide tests for the goodness of SDM for NDBD. Results of these studies as well as the $M^{0\nu}$ results for heavier ^{124}Sn, ^{130}Te and ^{136}Xe nuclei are in the process.

References

[1] Y. Fukuda *et al.* (Super-Kamiokande collaboration), *Phys. Rev. Lett.* **81** (1998) 1562.

[2] Q.R. Ahmad *et al.* (SNO collaboration), *Phys. Rev. Lett.* **87** (2001) 071301.

[3] Q.R. Ahmad *et al.* (SNO collaboration), *Phys. Rev. Lett.* **89** (2002) 011301.

[4] M. Auger *et al.*, *Phys. Rev. Lett.* **109** (2012) 032505.

[5] A. Gando *et al.*, *Phys. Rev. Lett.* **110** (2013) 062502.

[6] M. Agostini *et al.*, *Phys. Rev. Lett.* **111** (2013) 122503.

[7] F. Boehm and P. Vogel, *Physics of Massive Neutrinos*, Cambridge University Press, Cambridge (1992).

[8] J. Barea, J. Kotila, and F. Iachello, *Phys. Rev. C* **87** (2013) 014315.

[9] J. Suhonen and O. Civitarese, *J. Phys. G: Nucl. Part. Phys* **39** (2012) 124005.

[10] R. Sahu and V.K.B. Kota, *Int. J. Mod. Phys. E* **24** (2015) 1550022.

[11] V.K.B. Kota and R.U. Haq, *Spectral Distributions in Nuclei and Statistical Spectroscopy*, World Scientific, Singapore (2010).

[12] V.K.B. Kota, *Embedded Random Matrix Ensembles in Quantum Physics*, Lecture Notes in Physics 884, Springer, Heidelberg (2014).

[13] J.B. French, V.K.B. Kota, A. Pandey, and S. Tomsovic, *Ann. Phys. (N.Y.)* **181** (1988) 235.

[14] V.K.B. Kota and D. Majumdar, *Z. Phys. A* **351** (1995) 377.

[15] V.K.B. Kota and Manan Vyas, *Ann. Phys. (N.Y.)* **359** (2015) 252.

[16] M. Kortelainen, O. Civitarese, J. Suhonen, and J. Toivanen, *Phys. Lett. B* **647** (2007) 128.

[17] F.S. Chang, J.B. French, and T.H. Thio, *Ann. Phys. (N.Y.)* **66** (1971) 137.

[18] R.A. Sen'kov, M. Horoi, and V.G. Zelevinsky, *Comp. Phys. Comm.* **184** (2013) 215.

[19] R.A. Sen'kov and V.G. Zelevinsky, *Phys. Rev. C* **93** (2016) 064304.

[20] V.K.B. Kota and D. Majumdar, *Nucl. Phys. A* **604** (1996) 129.

[21] J.B. French, S. Rab, J.F. Smith, R.U. Haq, and V.K.B. Kota, *Can. J. Phys.* **84** (2006) 677.

[22] R. A. Sen'kov, M. Horoi, and B. A. Brown, *Phys. Rev. C* **89** (2014) 054304.

[23] A. S. Barabash, arXiv:nucl-ex/1112.1784

[24] M. Honma, T. Otsuka, T. Mizusaki, and M. Hjorth-Jensen, *Phys. Rev. C* **80** (2009) 064323.

[25] K. Kar, *J. Phys. G: Nucl. Part. Phys.* **40** (2013) 015105.

[26] ENSDF Data Base, Brookhaven National Laboratory, USA, `http://www.nndc.bnl.gov/ensdf/index.jsp`.

[27] J. Menendez, A. Poves, E. Caurier, and F. Nowacki, *Nucl. Phys. A* **818** (2009) 139.

[28] I. Abt *et al.*, arXiv:hep-ex/0404039; N. Abgrall *et al.*, *Adv. High Energy Phys.* **2014** (2014) 365432.

[29] R. A. Sen'kov and M. Horoi, *Phys. Rev. C* **93** (2016) 044334.

[30] V.K.B. Kota and V. Potbhare, *Nucl. Phys. A* **331** (1979) 93; S. Sarkar, K. Kar, and V.K.B. Kota, *Phys. Rev. C* **36** (1987) 2700.