Determinant of the neutrino mass is one of the most important subjects in physics because of its abundance in the universe and the necessity of extension of the standard theory. One of the promising, but not easy at all, few methods to determine the effective neutrino mass is to use the neutrinoless double-$\beta$ decay. This is a decay occurring if the neutrino is a Majorana particle, and not yet observed. The effective neutrino mass can be determined if the half-life is measured, and the transition matrix element is calculated.

The transition matrix element consists of the factor arising from the emitted electrons and nuclear matrix element. The latter is more difficult to calculate accurately than the former because the accurate nuclear wave functions are necessary. Currently the nuclear matrix elements are distributed in a range of factor of $2 - 3$ depending on the method of approximation.

In my study the nuclear matrix elements of neutrinoless and two-neutrino double-$\beta$ decays are calculated using the quasiparticle random-phase approximation (QRPA), and the consistency of the method is examined in the example of $^{48}$Ca. There are two major checkpoints. The QRPA approach uses two sets of intermediate states obtained on the basis of the initial and final states. The question on the validity of this approach is possible for the two-neutrino double-$\beta$ decay because the calculation needs the energies of the intermediate states explicitly (the neutrinoless one can avoid this thanks to an approximation valid for the neutrino-exchange interaction). First, I show that the two results using the different sets of the intermediate-state energies are very close, as long as the appropriate strength of the isoscalar proton-neutron pairing interaction is used. The half-life data of the two-neutrino double-$\beta$ decay is used for determining the effective axial-vector current coupling.

Second, I discuss the experimental data of the spin-flip charge exchange transition-strength functions obtained by the $(p,n)$ and $(n,p)$ reactions. The cross sections of these reactions and the above nuclear matrix element share the charge-exchange transition density. Therefore, the reproduction of this data is an important checkpoint of the calculation of the nuclear matrix element.