NUCLEAR THEORY, Vol. 37 (2018)
eds. M. Gaidarov, N. Minkov, Heron Press, Sofia

$\gamma-$Rigid Triaxial Nuclei in the Presence of a Minimal Length via a Quantum Perturbation Method

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Abstract. In this work, we derive a closed solution of the Shrödinger equation for Bohr Hamiltonien within the minimal length formalism. This formalism is inspired by Heisenberg algebra and a generalized uncertainty principle (GUP), applied to the geometrical collective Bohr–Mottelson model (BMM) of nuclei by means of deformed canonical commutation relation and the Pauli-Podolsky prescription. The problem is solved by means conjointly of asymptotic iteration method (AIM) and a quantum perturbation method (QPM) for transitional nuclei near the critical point symmetry Z(4) corresponding to phase transition from prolate to $\gamma - rigid$ triaxial shape. A scaled Davidson potential is used as a restoring potential in order to get physical minimum. The agreement between the obtained theoretical results and the experimental data is very satisfactory.

1 Introduction

Critical point symmetries in nuclear structure are recently receiving considerable attention [1,2] since they provide parameter-free solutions. The pioneering ones amid them were E(5) [1,3], X(5) [4] and Z(5) [5] corresponding to shape phase transitions from U(5) to O(6), U(5) to SU(3) and from axial to triaxial shapes respectively, with the recent addition of Y(5) [2] related to the transition from prolate to oblate shape . Later, a γ-rigid (with $\gamma = 0$) version of X(5), called X(3), has been introduced [6]. In the same way, other CPS have been developped like for example Z(4) (with $\gamma = \pi/6$) the gamma rigid version of Z(5) corresponding to shape phase transitions from prolate to triaxial symmetry [7–10]. From a structural point of view, the collective Bohr Mottelson represents a sound frame work to describe many properties of the quadrupole collective dynamics in nuclei [3]. Its formulation has the ability to describe both rotational and vibrational modes. On the other hand, recently, a great interest has been consecrated to the quantum mechanical problems related to a generalized modified commutation relations involving a minimal length or generalized uncertainty principle [11,12].

In the present work we focuse on the study of the quadrupole collective states in γ-rigid case, by modifying Davydov-Chaban Hamiltonian in the framework of the minimal length formalism [13] with Davidson potential [14] for β-vibrations. The model is conventionally called Z(4)-D-ML. The organization of this paper is as follows: in Section 2, we present the Z(4)-D-ML model with the quantum perturbation method which requires the study of the model in the absence and in the presence of the minimal length, presented in Sections 3 and 4 respectively. Finally, Section 5 is devoted to the numerical results and brief discussion for energy spectrum of some triaxial-rigid nuclei, while Section 6 contains our conclusions.

2 Z(4)-D-ML with the Quantum Perturbation Method

In the Z(4) model the γ variable is frozen to $\gamma = \pi/6$ and only four variables are involved; three Euler angles $(\theta_1, \theta_2, \theta_3)$, which obviously define the orientation of the intrinsic principal axes in the laboratory frame and the deformation parameter. So, in the frame of the Bohr-Mottelson model [3,15], the corresponding eigenvalue problem reduced to that of the Davydov-Chaban hamiltonian [7]. Therefore, we aimed to study the minimal length effect on energy spectrum in the context of γ-rigid nuclei Z[4].

$$H_{DC} = -\frac{\hbar^2}{2B_m}\left[\frac{1}{\beta^3}\frac{\partial}{\partial\beta}\beta^3\frac{\partial}{\partial\beta} - \frac{1}{4\beta}\sum_{k=1}^{3}\frac{Q_k^2}{\sin^2(\gamma - \frac{2\pi}{3}k)}\right] + U(\beta), \tag{1}$$

where β and γ are the usual collective coordinates [3], $Q_k (k = 1, 2, 3)$ are the components of angular momentum and B_m is the mass parameter. In this Hamiltonian γ is treated as a parameter and not as a variable.

By employing the mathematical formulation, including the minimal length concept, presented in the original paper [13], the collective equation of eigenstates, up to the first order of α, is written as follows:

$$\left(-\frac{\hbar^2}{2B_m}\Delta + \frac{\alpha\hbar^4}{B_m}\Delta^2 + V(\beta) - E_{n,L}\right)\psi(\beta, \theta_i) = 0, \tag{2}$$

where

$$\Delta = \frac{1}{\beta^3}\frac{\partial}{\partial\beta}\beta^3\frac{\partial}{\partial\beta} - \frac{\Delta_\theta}{4\beta} \tag{3}$$

$$\Delta_\theta = \sum_{k=1}^{3}\frac{Q_k^2}{\sin^2(\gamma - \frac{2\pi}{3}k)} \tag{4}$$

and $\theta_i (i = 1, 2, 3)$ are the Euler angles. This equation can be simplifed by introducing an auxiliary wave function [13]:

$$\psi(\beta, \theta_i) = (1 + 2\alpha\hbar^2\Delta)\xi(\beta, \theta_i). \tag{5}$$

Thus, we obtain the following differential equation satisfied by $\xi(\beta, \theta_i)$

$$\left[(1 + 4B_m\alpha(E - V(\beta)))\Delta + \frac{2B_m}{\hbar^2}(E_{n,L} - V(\beta))\right]\xi(\beta, \theta_i) = 0. \tag{6}$$

This equation can be simplified by using the usual following factorization:

$$\xi(\beta, \theta_i) = \Phi(\beta)\chi(\theta_i). \tag{7}$$

The separation of variables leads to two equations: one depending only on the β variable; and the other depending on the γ and the Euler angles

$$\left[\frac{1}{\beta^3}\frac{\partial}{\partial\beta}\beta^3\frac{\partial}{\partial\beta} - \frac{\Lambda}{\beta^2} + \frac{2B_m}{\hbar^2}\bar{k}(E_{n,L}, \beta)\right]\Phi(\beta) = 0, \tag{8}$$

where

$$\bar{K}(E_{n,L}, \beta) = \left(\frac{E_n - V(\beta)}{1 + 4B_m\alpha(E_{n,L} - V(\beta)}\right) \tag{9}$$

n is the radial quantum number

$$\left[\frac{1}{4}\sum_{k=1}^{3}\frac{Q_k^2}{\sin^2(\gamma - \frac{2\pi}{3}k)} - \Lambda\right]\chi(\theta_i) = 0. \tag{10}$$

In the case of $\gamma = \pi/6$, the last equation takes the form [16]:

$$[\frac{1}{4}(Q_1^2 + 4Q_2^2 + 4Q_3^2) - \Lambda]\chi(\theta_i) = 0. \tag{11}$$

This equation has been solved by Meyer-ter-Vehn [16], the eigen functions being:

$$\chi(\theta_i) = \chi(\theta_i)^L_{\mu,\omega} = \sqrt{\frac{2L+1}{16\pi^2(1+\delta_\omega, 0)}}\left[D^L_{\mu,\omega}(\theta_i) + (-1)^L D^L_{\mu,-\omega}(\theta_i)\right]. \tag{12}$$

Here, $D^L_{\mu,\omega}(\theta_i)$ represents the Weigner function of the Euler angles, L are the eigenvalues of angular momentum, while ω and μ are the eigenvalues of the projections of angular momentum on the body-fixed x-axis and the laboratory fixed z-axis, respectively [16] with

$$\Lambda = \Lambda_{L,\omega} = L(L+1) - \frac{3}{4}\omega^2. \tag{13}$$

Thanks to the smallness of the parameter α, by expanding (9) in power series of α, one can obtain different order approximations of the standard model Z(4)-ML. At the first order approximation, as it has been done recently in [17], (9) becomes

$$\begin{aligned}\bar{K}(E_{n,L} - V(\beta)) &\approx (E_{n,L} - V(\beta))(1 - 4B_m\alpha(E_{n,L} - V(\beta))) \\ &= E_{n,L} - V(\beta) - 4B_m\alpha(E_{n,L} - V(\beta))^2. \end{aligned}\tag{14}$$

In what concerns the β degree of freedom, we will consider the Davidson potential chosen to be of the following form:

$$V(\beta) = a\beta^2 + \frac{b}{\beta^2}, \quad \beta_0 = (\frac{b}{a})^{\frac{1}{4}}, \tag{15}$$

where a and b are two free scaling parameters, and β_0 represents the position of the minimum of the potential. The special case of $b = 0$ ($\beta_0 = 0$) corresponds to the simple harmonic oscillator.

The differential equation (8) was solved exactly, with an infinite square well like potential, within the standard model [21] but it is not soluble analytically for the Davidson-type potential. However, the quantum perturbation theory one of its familiar forms, dubbed the quantum perturbation method (QPM), is used to obtain approximate solutions for all values of angular momentum L [17].

3 Z(4) Model with Davidson Potential Z(4)-D ($\alpha = 0$)

It is preferable to write the equation in a Schrödinger picture. This is realized by changing the wave function as $\Phi(\beta) = \beta^{-\frac{3}{2}} f(\beta)$. However one obtains an equation which resembles the radial Schrödinger equation for an isotropic harmonic oscillator acting in four-dimensional space:

$$\frac{d^2}{d\beta^2} f(\beta) + \left[\frac{2B_m E_{n,L}}{\hbar^2} - \frac{2B_m a}{\hbar^2}\beta^2 - \frac{L(L+1) + \frac{3}{4}(1-\alpha^2) + \frac{2bB_m}{\hbar^2}}{\beta^2}\right] f(\beta) = 0. \tag{16}$$

We define

$$\epsilon = \frac{2B_m E_{n,L}}{\hbar^2}, \qquad \omega = \frac{2B_m a}{\hbar^2}$$

and

$$\vartheta_{l,b}(\vartheta_{l,b} + 1) = L(L+1) + \frac{3}{4}(1-\alpha^2) + \frac{2bB_m}{\hbar^2}. \tag{17}$$

However, one obtains an equation which resembles to the Goldman and Krivchenkov Hamiltonian [18]

$$\frac{d^2}{d\beta^2} f(\beta) + \left[\epsilon - \omega\beta^2 - \frac{\vartheta_{l,b}(\vartheta_{l,b}+1)}{\beta^2}\right] f(\beta) = 0. \tag{18}$$

To solve this differential equation via the asymptotic iteration method (AIM) [18], we propose the following ansatz [18]:

$$f(\beta) = \beta^{1+\vartheta_{l,b}} + e^{-\frac{\sqrt{\omega}}{2}\beta^2}\Theta(\beta). \tag{19}$$

Thus we obtain,

$$\frac{d^2\Theta(\beta)}{d\beta^2} + (\frac{2p}{\beta} - 4q\beta)\frac{d\Theta(\beta)}{d\beta} + [\epsilon - 2q(1+2p)]\Theta(\beta) = 0, \qquad (20)$$

where $q = \frac{\sqrt{\omega}}{2}$ and $p = 1 + \vartheta_{l,b}$.

After calculating λ_0 and s_0, by means of the recurrence relations [18], we get the generalized formula of the reduced energy from the roots of the quantization condition

$$\epsilon = q[2 + 4p + 8n]; \quad n = 1, 2, \ldots, \qquad (21)$$

from which, we obtain the energy spectrum

$$E_{n,L} = \frac{\hbar^2}{2B_m}\epsilon = \sqrt{\frac{\hbar^2}{2B_m}}a[3 + 4n_\beta + 2\vartheta_{l,b}]. \qquad (22)$$

From equation (17), we get $\vartheta_{l,b}$ as a function of the total angular momentum L and the parameter b

$$\vartheta_{l,b} = -\frac{1}{2} + \frac{1}{2}\sqrt{4L(L+1) - 3\alpha^2 + 8b + 4}\,. \qquad (23)$$

The physical solutions to (8) are obtained as

$$\Phi(\beta) = N_{n\beta,L}\beta^{-\frac{3}{2}+p}e^{-q\beta^2}L_{n\beta}^{p-\frac{1}{2}}(2q\beta^2)\,, \qquad (24)$$

where $L_{n\beta}^{p-\frac{1}{2}}(2q\beta^2)$ denotes the associated Laguerre polynomials and $N_{n\beta,L}$ is a normalization constant.

4 Z(4) Model with Davidson Potential via a Minimal Length (Z(4)-D-ML)

Here, we treat the additional term $\frac{\alpha\hbar^4}{B_m}\Delta^2$ shown in equation (2) as a perturbation and then estimate its effect on the energy spectrum up to the first order of the perturbation theory. Hence, the energy spectrum can be written as

$$E_{n,L} = E^0_{n,L} + \Delta E_{n,L}, \qquad (25)$$

where $E^0_{n,L}$ is the unperturbed energy spectrum, and $\Delta E_{n,L}$ the correction induced by the minimal length, given by

$$\Delta E_{n,L} = \frac{\alpha\hbar^4}{B_m}\langle\psi^0(\beta,\theta_i) \mid \Delta^2 \mid \psi^0(\beta,\theta_i)\rangle, \qquad (26)$$

where $\psi^0(\beta,\theta_i)$ are the eigenfunctions, solutions to the ordinary Schrödinger equation ($\alpha = 0$).

So, the energy spectrum can be expressed as [17]

$$\Delta E_{n,L} = 4B_m\alpha\Big[(E^0_{n,L})^2 - 2E^0_{n,L}\langle\psi^0 \mid V(\beta) \mid \psi^0\rangle + \langle\psi^0 \mid V(\beta)^2 \mid \psi^0\rangle\Big]. \quad (27)$$

After substituting the Davidson potential (15), one obtains

$$\Delta E_{n,L} = 4B_m\alpha\Big[(E^0_{n,L})^2 + 2ab - 2E^0_{n,L}(a\overline{\beta^2} + b\overline{\beta^{-2}}) + (a^2\overline{\beta^4} + b^2\overline{\beta^{-4}})\Big], \quad (28)$$

where $\overline{\beta^i}$ $(i = 2, -2, 4, -4)$ are expressed as follows:

$$\begin{cases} \overline{\beta^2} = \dfrac{4n + 2\vartheta_{l,b} + 3}{4q} \\ \overline{\beta^{-2}} = \dfrac{4q}{2\vartheta_{l,b} + 1} \\ \overline{\beta^4} = \dfrac{4\vartheta^2_{l,b} + 24n\vartheta_{l,b} + 24n^2 + 16\vartheta_{l,b} + 36n + 15}{16q^2} \\ \overline{\beta^{-4}} = \dfrac{16q^2(4n + 2\vartheta_{l,b} + 3)}{(2\vartheta_{l,b} + 3)(4\vartheta^2_{l,b} - 1)} \end{cases} \quad (29)$$

Details of $\overline{\beta^i}$ calculations are given in [17].

5 Numerical Examination

The model established in this work, called Z(4)-D-ML, is adequate for description of γ-rigid nuclei for which the γ parameter is fixed to $\gamma = \pi/6$. Basically, the energy levels of the ground state band as well as of the vibrational bands are characterized by the principal quantum number n_β and n_ω, respectively. with n_ω is the wobbling quantum number [16, 19] $n_\omega = L - \omega$. We briefly recall a few interesting low-lying bands which are classified by the quantum numbers n_β and n_ω

- The ground state band (gsb) with $n_\beta = 0$ and $n_\omega = 0$
- The β band with $n_\beta = 1$ and $n_\omega = 0$
- The γ band composed by the even L levels with $n_\beta = 0$ and $n_\omega = 2$ and the odd L levels with $n_\beta = 0$ and $n_\omega = 1$
- For our subsequent calculations, we define the energy ratios as:

$$R(n_\beta, L, n_\omega) = \frac{E_{n_\beta,L,n_\omega} - E_{0,0,0}}{E_{0,2,0} - E_{0,0,0}}$$

- The mentioned results are thus found to have the smallest deviations from the experimental data [20], evaluated by the quality measure

$$\sigma = \sqrt{(\sum_{i=1}^{N}(E_i^{exp} - E_i^{th})^2)/N},$$

where N is the maximum number of levels.

We have treated 32 nuclei among which are depicted in Figure 1 those with good results.

The comparison between Z(4)-D-ML theoretical predictions and experimental data [20] of selected candidates regarding energy levels is visualized schematically in Figure 1. The agreement with experiment is very good for the ground state band and β band, despite the fact that there is not much experimental data

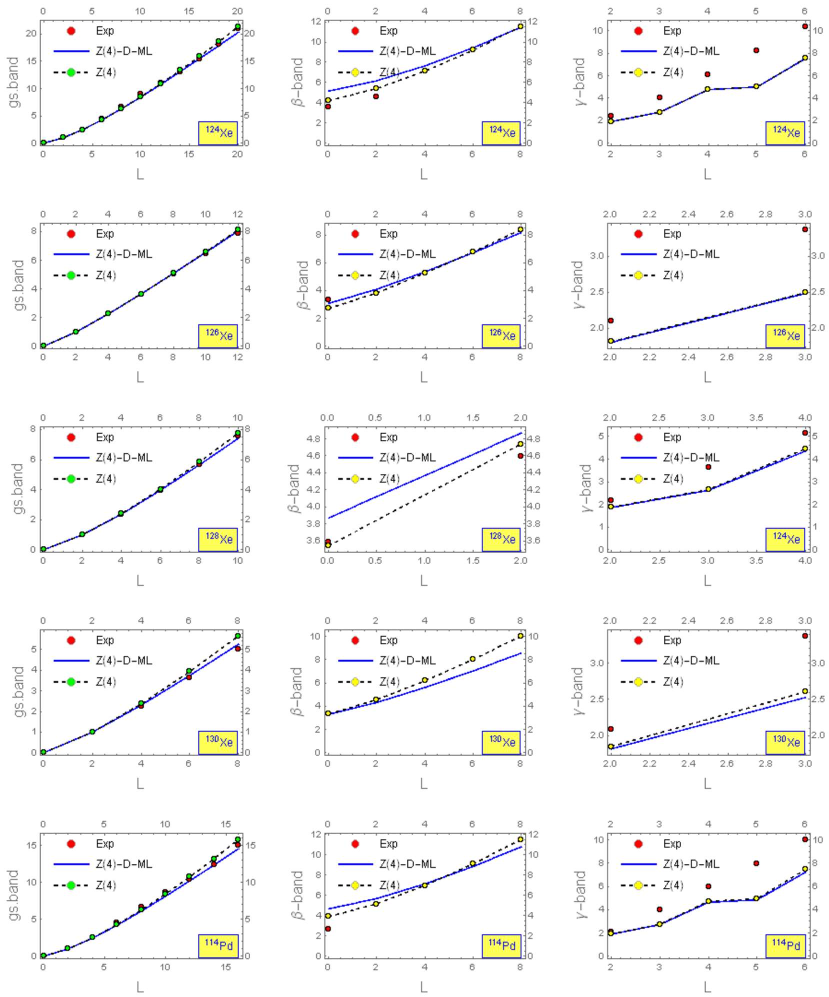

Figure 1. Comparison of the Z(4)-ML-D predictions for (normalized) energy levels to experimental data for ^{124}Xe, ^{126}Xe, ^{128}Xe, ^{130}Xe, ^{114}Pd.

especially for the γ band of these studied nuclei. As a result, one concludes that the Z(4)- D-ML is more suitable for describing the structural properties of nuclei having a structure in vicinity of the Z(4) limit.

Table 1. Standard deviation between experimental and theoretical results

Nuclei	σ_D	σ_{D-ML}	β_0
^{124}Xe	0.588	0.44	0.82
^{126}Xe	0.36	0.36	0.61
^{128}Xe	0.44	0.40	0.69
^{130}Xe	0.35	0.35	0.62
^{114}Pd	0.84	0.63	0.78

6 Conclusion

The idea of Z(4)-ML is already used and presented with the square well potential [21], but this time it was used with the Davidson potential. However, the Hamiltonian of the system is not soluble analytically for a potential other than the square well. In order to overcome such a difficulty, in the present work we used, a quantum perturbation method (QPM), to obtain approximate solutions for all values of angular momentum L. Therefore, closed-form analytical formula for the energy of the ground and vibrational bands was derived for triaxial γ-rigid nuclei within Davidson potential. Our results indicate a better agreement with the experimental values, and reproduced well the best Z(4) candidate nuclei already obtained in the Xe region around $A = 130$ including the new one ^{114}Pd.

Acknowledgements

S. Ait Elkorchi would like to thank the organizing committee for the hospitality and the wonderful scientific meet. Also, she acknowledges the financial support (Type A) of Cadi Ayyad University.

References

[1] F. Iachello, *Phys. Rev. Lett.* **85** (2000) 3580.
[2] F. Iachello, *Phys. Rev. Lett.* 91 (2003) 132502.
[3] A. Bohr, *Mat. Fys. Medd. Dan. Vid. Selsk.* **26** (1952) 14.
[4] F. Iachello, *Phys. Rev. Lett.* **87** (2001) 052502.
[5] D. Bonatsos, D. Lenis, D. Petrellis, P.A. Terziev, *Phys. Lett. B* **588** (2004) 172.
[6] D. Bonatsos, D. Lenis, D. Petrellis, P. Terziev and I. Yigitoglu, *Phys. Lett. B* **632** (2006) 238.
[7] A.S. Davydov, A.A. Chaban, *Nucl. Phys.* **20** (1960) 499.
[8] D. Bonatsos, D. Lenis, D. Petrellis, P.A. Terziev, I. Yigitoglu, *Phys. Lett. B* **621** (2005) 102.

[9] A.S. Davydov and G.F. Filippov, *Nucl. Phys*. **8** (1958) 237.
[10] A.S. Davydov and V.S. Rostovsky, *Nucl. Phys*. **12** (1959) 58.
[11] A. Kempf, *J. Math. Phys.* **35** (1994) 4483.
[12] F. Brau, *J. Phys. A: Math. Gen.* **32** (1999) 7691.
[13] M. Chabab, A. El Batoul, A. Lahbas and M. Oulne, *Phys. Lett. B* **758** (2016) 212.
[14] P. Davidson, *Proc. R. Soc. London Ser. A* **135** (1932) 459472.
[15] A. Bohr, B.R. Mottelson, *Mat. Fys. Medd. Dan. Vid. Selsk*. **27** (1953) 16.
[16] J. Meyer-ter-Vehn, *Nucl. Phys. A* **249** (1975) 111.
[17] M. Chabab, A. El Batoul, A. Lahbas, M. Oulne, *Ann. Phys.* **392** (2018) 142.
[18] H. Ciftci, R.L. Hall and N. Saad, *J. Phys. A: Math.* **36** (2003) 11807.
[19] A. Bohr, B.R. Mottelson, "*Nuclear Structure*", vol. II (Benjamin, New York, 1975).
[20] Nuclear Data Sheets (http://nndc.bnl.gov/).
[21] M. Chabab, A. El Batoul, A. Lahbas, M. Oulne, In: *Nuclear Theory* Vol. 36 (2017), edited by M. Gaidarov, N. Minkov, Heron Press, Sofia, pp. 215-223.