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Comparison Between Coulomb and Hulthèn Potentials Within Bohr Hamiltonian for γ-Rigid Nuclei in the Presence of Minimal Length

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Abstract. In this work we solve the Schrödinger equation for Bohr Hamiltonian with Coulomb and Hulthén potentials within the formalism of minimal length in order to obtain analytical expressions for the energy eigenvalues and eigenfunctions by means of asymptotic iteration method. The obtained formulas of the energy spectrum and wave functions, are used to calculate excitation energies and transition rates of γ-rigid nuclei and compared with the experimental data at the shape phase critical point X(3) in nuclei.

1 Introduction

Several analytical solutions of the Bohr Hamiltonian with different model potentials have been proposed. On the other hand, this problem is related to the evolution of Critical Point Symmetries concept. For example, the symmetry E (5) [1] describes the second-order phase transition between spherical and γ-unstable nuclei, while the transition from vibratory to axially symmetric nuclei is described by symmetry X (5) [2] and X(3) [3] which is a special case of this latter in which γ is fixed to γ=0. This model has been developed with the introduction of the concept of minimal length [4]. In this context, different model potentials have been used such as infinite Square Well (ISW) [5], the harmonic oscillator [6], the sextic potential [7] and the Davidson one within X(3) symmetry.

In the present work we focused on the study of the Bohr Hamiltonian in the presence of a minimal length in X(3) model with two known potentials, namely: Hulthén and coulomb, where we have obtained the expressions of eigenvalues and wave functions by means of the asymptotic iteration method (AIM) [8, 9]. Such a useful method is efficient to solve many similar problems [10, 11].

2 Formulation of the Model

The Bohr Hamiltonian in the presence of a minimal length is given by [4]

$$H = -\frac{\hbar^2}{2B_m}\Delta + \frac{\alpha\hbar^4}{2B_m}\Delta^2 + V(\beta) \tag{1}$$

with

$$\Delta = \frac{1}{\beta^2}\frac{\partial}{\partial\beta}\beta^2\frac{\partial}{\partial\beta} + \frac{1}{3\beta^2}\Delta_\Omega, \tag{2}$$

where Δ_Ω is the angular part of the Laplace operator

$$\Delta_\Omega = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin\theta^2}\frac{\partial^2}{\partial\phi^2}. \tag{3}$$

The corresponding deformed Schrödinger equation to the first order in α reads as

$$\left[-\frac{\hbar^2}{2B_m}\Delta + \frac{\alpha\hbar^4}{2B_m}\Delta^2 + V(\beta) - E\right]\psi(\beta,\theta,\phi) = 0. \tag{4}$$

By introducing an auxiliary wave function

$$\psi(\beta,\theta,\phi) = \left[1 - 2\alpha\hbar^2\Delta\right]\phi(\beta,\theta,\phi), \tag{5}$$

we obtain the following differential equation satisfied by ϕ

$$\left[(1 + 4B_m\alpha(E - V(\beta)))\Delta + \frac{2B_m}{\hbar^2}(E - V(\beta))\right]\phi(\beta,\theta,\phi) = 0 \tag{6}$$

By considering the wave function as

$$\phi(\beta,\theta,\phi) = \xi(\beta)Y_{LM}(\theta,\phi)$$

and

$$\Delta_\Omega Y_{LM}(\theta,\phi) = -L(L+1)Y_{LM}(\theta,\phi)$$

Eq. (6) transforms into [4]

$$\frac{1}{\beta^2}\frac{\partial}{\partial\beta}\beta^2\frac{\partial}{\partial\beta}\xi(\beta) + \left(\frac{-\Lambda}{\beta^2} + \frac{2B_m}{\hbar^2}((E - V(\beta)) - 4B\alpha(E - V(\beta))^2)\right)\xi(\beta), \tag{7}$$

where $V(\beta)$ is

- The Coulomb potential:

$$V(\beta) = \frac{c}{\beta}\,; \tag{8}$$

- The Hulthén potential:

$$V(\beta) = \frac{e^{-\delta\beta}}{e^{-\delta\beta} - 1}\,. \tag{9}$$

3 Energy Spectrum

3.1 Hulthén potential

Using the new variable $y = e^{-\delta\beta}$, Eq. (7) becomes

$$\frac{d^2}{dy^2}\xi(y) + \frac{1}{y}\frac{d}{dy}\xi(y) + \frac{1}{\delta^2 y^2}\Big(- \Lambda\delta^2 \frac{y}{(y-1)^2} + \frac{2B_m}{\hbar^2}((E - \frac{y}{(y-1)} - 4B\alpha(E - \frac{y}{(y-1)})^2\Big)\xi(y). \quad (10)$$

In order to apply AIM, we consider the following ansatz:

$$\xi(y) = y^{\nu}(1-y)^{\mu}\chi(y) \quad (11)$$

with

$$\nu = \frac{\sqrt{-2E}}{\delta} \quad \text{and} \quad \mu = \frac{1}{2}\left(1 + \sqrt{4\Lambda + 1 + \frac{32\alpha}{\delta^2}}\right).$$

Using the AIM, we obtain the energy spectrum in the following form:

$$E = -\frac{1}{8}\left[\frac{-\delta^2(\mu+n_\beta)^2 + 8\alpha + 1}{\delta(\mu+n_\beta)}\right]^2. \quad (12)$$

3.2 Coulomb potential

By substituting the following ansatz $\xi(\beta) = \beta^{\mu}e^{\nu\beta}\xi(\beta)$ in Eq. (7), we get

$$\frac{d^2}{d\beta^2}\xi(\beta) + \Big[\frac{2\mu + 2\nu\beta + 2}{\beta}\Big]\frac{d}{d\beta}\xi(\beta) + \Big[\frac{16\alpha c E_0 + 2\mu\nu - 2c + 2\nu}{\beta}\Big]\xi(\beta) \quad (13)$$

with

$$\mu = -\frac{1}{2} + \frac{1}{2}\sqrt{32\alpha c^2 + 4\Lambda + 1} \quad \text{and} \quad \nu = -\sqrt{8\alpha E_0^2 - 2E}\,.$$

Applying the AIM, we obtain the energy spectrum as

$$E = \frac{-2c^2}{G}((8\alpha E_0 - 1)^2 + 4\alpha E_0^2) \quad (14)$$

with

$$G = 4n_\beta^2 + 4n_\beta + 2 + \frac{4}{3}L(L+1) + 32\alpha c^2 + 4(2n_\beta+1)\sqrt{\frac{1}{4} + \frac{L(L+1)}{3} + 8\alpha c^2}$$

and

$$E_0 = \frac{-2c^2}{\left[(2n_\beta + 1) + 2\sqrt{\frac{1}{4} + \frac{L(L+1)}{3}}\right]^2}$$

4 Wave functions

4.1 Hulthén

The wave function is written in terms of Hypergeometric functions

$$\xi = Ne^{-\delta\beta\nu}(1-e^{-\delta\beta\mu})_2F_1[-n, 2\mu+2+2\nu+n, 2\nu+1, e^{-\delta\beta}], \tag{15}$$

where N is a normalization constant [12]

$$N = \left[\frac{\mu+n}{2\delta\nu(\nu+\mu+n)}\right]^{-0.5}\left[\frac{(\Gamma(2\nu+1)\Gamma(n+1))^2\Gamma(2\mu+n)}{n!\Gamma(2\nu+n+1)\Gamma(2\nu+2\mu+n)}\right]^{-0.5}. \tag{16}$$

4.2 Coulomb

The wave function in this case is written in terms of Laguerre polynomials

$$\xi(\beta) = Ne^{-\nu\beta}\beta^{\mu} KummerM[-n, 2\mu+2, 2\nu\beta] \tag{17}$$

with

$$N = \left[\left(\frac{1}{2\nu}\right)^{2\mu+3}\frac{\Gamma(n+2\mu+2)(2n+2\mu+2)}{n!(LaguerreL[n,2\mu+1,0])^2}\right]^{-0.5}. \tag{18}$$

5 Transition rates B(E2)

The general expression for the quadrupole transition operator is [13]

$$T^{E2}_M = t\beta\Big[D^{2*}_{M,0}(\theta_i)cos(\gamma) + \frac{1}{\sqrt{2}}[D^{2*}_{M,2}(\theta_i) + D^{2*}_{M,-2}(\theta_i)\sin(\gamma)]\Big], \tag{19}$$

where t denotes a scalar factor and $D^{2*}_{M,2}(\theta_i)$ is the Winger functions of Euler angles.

The B(E2) transition rates are given by [3]

$$B(E2, nLn_\gamma K \to n'L'n'_\gamma K) = t^2\langle L2L'|K, K'-K, K'\rangle^2 I^2_{n,L,n',L'}, \tag{20}$$

where $\langle L2L'|K, K'-K, K'\rangle$ are Clebsch-Gordan coefficients and

$$I_{n,L,n',L'} = \int_0^\infty \beta\xi_{n,L}(\beta)\xi_{n',L'}(\beta)\beta^2 d\beta\,. \tag{21}$$

6 Numerical results

6.1 Spectra of γ-rigid nuclei

The formulas of the energy spectrum, obtained by the equations 12 and 14, are used to calculate the excitation energies of γ-rigid nuclei. The energy spectrum of Coulomb potential depends on two parameters (α, c), while in the Hulthén potential, it depends on (α, δ). All these parameters have been set by fitting the excitation energies normalized to the energy of the first excited state $E(2_1^+)$. We evaluate the root mean square (rms) deviation between theoretical values and the

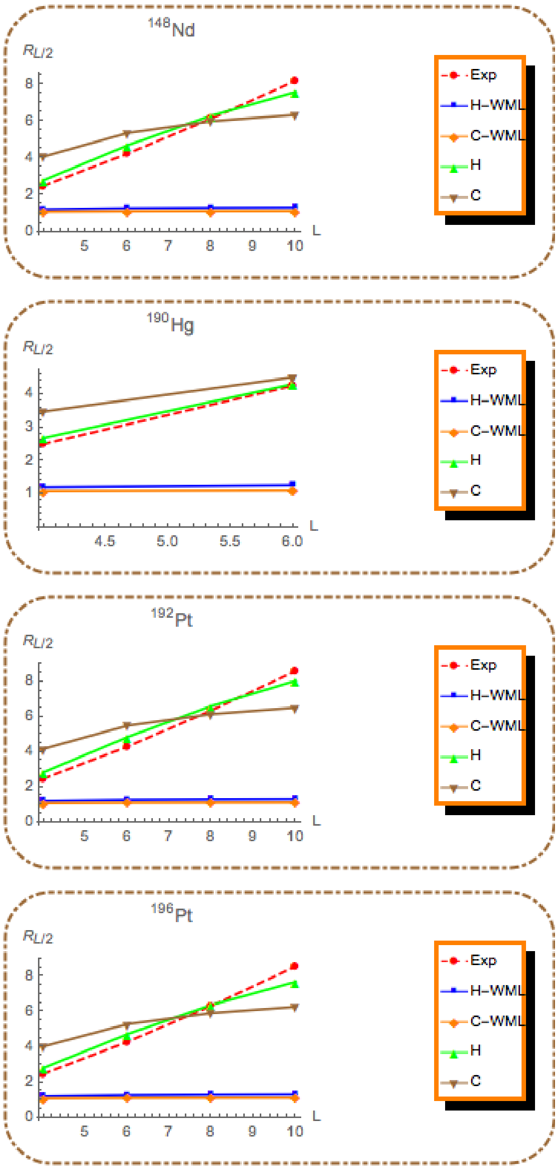

Figure 1. The energy ratios in the absence of ML and in its presence

experimental data by

$$\sigma = \sqrt{\frac{\sum_{i=1}^{m}(E_i(\text{exp})) - E_i(\text{th}))^2}{(m-1)E(2_1^+)^2}} \tag{22}$$

where m is the number of states, while E_i(exp) and E_i(th) represent the theoretical and experimental energies of the i^{th} level, respectively. $E(2_1^+)$ is the energy of the first excited level of the ground state band.

From Eq. (9) and (8), one can see that both potentials have mathematically similar behaviors. If we give the same value to the parameter c in Coulomb potential (Eq. (8)) and δ in the Hulthén one (Eq. (9)), we get overcome curves. Figure (1) shows that in the absence of minimal length case, the obtained results for energy ratios with both potentials are identical for all even-even nuclei, while in its presence, the calculated energy ratios $R_{L/2}$ with Hulthén potential are

Table 1. The comparison of the obtained results by the two equations: (12) and (14), for the ground state band $(n = 0)$ and the β band $(n = 1)$ with the experimental data [14]. The values of free parameters is also shown

Nucleus		$R_{0,4}$	$R_{0,6}$	$R_{0,8}$	$R_{0,10}$	$R_{1,0}$	$R_{1,2}$	$R_{1,4}$	$R_{1,6}$	α	δ	c	σ
^{104}Ru	Exp	2.48	4.35	6.48	8.69	2.76	4.23	5.81					
	H	2.82	4.78	6.49	7.85	3.91	4.51	5.62		0.00003	0.003		0.63
	C	4.121	5.42	6.06	6.42	4.11	4.63	5.51		0.58		-1.31	1.36
^{120}Xe	Exp	2.44	4.23	6.34	8.77	2.82	3.95	5.31					
	H	2.81	4.75	6.43	7.76	3.89	4.48	5.58		0.00003	0.003		0.70
	C	4.07	5.35	5.98	6.33	4.06	4.58	5.44		0.58		-1.31	1.43
^{122}Xe	Exp	2.50	4.43	6.69	9.18	3.47	4.51						
	H	2.86	4.94	6.81	8.34	4.18	4.79			0.00004	0.003		0.58
	C	4.29	5.67	6.35	6.72	4.29	4.84			0.58		-1.31	1.53
^{124}Xe	Exp	2.48	4.37	6.58	8.96	3.58	4.60	5.69					
	H	2.85	4.89	6.71	8.19	4.10	4.70	5.86		0.00003	0.003		0.47
	C	4.25	5.61	6.28	6.65	4.25	4.79	5.71		0.58		-1.31	1.32
^{148}Nd	Exp	2.49	4.24	6.15	8.19	3.04	3.88	5.32	7.12				
	H	2.79	4.68	6.30	7.58	3.77	4.36	5.44	6.63	0.00003	0.003		0.49
	C	4.07	5.36	5.99	6.34	4.07	4.58	5.45	6.00	0.58		-1.31	1.31
^{150}Sm	Exp	2.32	3.83	5.50	7.29	2.22	3.13	4.34	6.31				
	H	2.66	4.27	5.54	6.47	3.23	3.78	4.73	5.70	0.00003	0.004		0.64
	C	3.58	4.66	5.19	5.48	3.55	4.00	4.73	5.19	0.58		-1.30	1.17
^{152}Gd	Exp	2.19	3.57	5.07	6.68	1.79	2.70	3.72	4.85				
	H	2.53	3.88	4.86	5.54	2.80	3.31	4.15	4.93	0.00003	0.005		0.67
	C	3.17	4.08	4.52	4.77	3.12	3.52	4.14	4.53	0.41		-1.52	1.06
^{172}Os	Exp	2.66	4.63	6.70	8.89	3.33	3.56	5.00	6.81				
	H	2.81	4.76	6.45	7.79	3.88	4.47	5.58	6.82	0.00003	0.003		0.63
	C	4.19	5.52	6.17	6.54	4.18	4.72	5.61	6.18	0.58		-1.31	1.29
^{190}Hg	Exp	2.50	4.26			3.07	3.77	4.74	6.03				
	H	2.68	4.31			3.44	3.88	4.58	5.02	0.00004	0.004		0.16
	C	3.48	4.51			3.48	4.51	5.01	5.30	0.95		-1.02	0.66
^{192}Pt	Exp	2.48	4.31	6.38	8.62	3.78	4.55						
	H	2.84	4.85	6.629	8.06	4.02	4.63			0.00003	0.003		0.41
	C	4.18	5.50	6.16	6.52	4.17	4.71			0.58		-1.31	1.33
^{196}Pt	Exp	2.47	4.29	6.33	8.56	3.19	3.83						
	H	2.80	4.72	6.37	7.68	3.82	4.41			0.00002	0.003		0.60
	C	4.03	5.29	5.91	6.26	4.02	4.53			0.58		-1.31	1.41

Table 2. The comparison of the present model with experimental data, for theoretical predictions calculated with the two potentials: Hulthén and Coulomb, for $B(E2)$ transition rates [15]

Nucleus		$\frac{4_1\to 2_1}{2_1\to 0_1}$	$\frac{6_1\to 4_1}{2_1\to 0_1}$	$\frac{8_1\to 6_1}{2_1\to 0_1}$	$\frac{10_1\to 8_1}{2_1\to 0_1}$	$\frac{0_\beta\to 2_1}{2_1\to 0_1}$	$\frac{2_\beta\to 2_1}{2_1\to 0_1}$	$\frac{2_\beta\to 4_1}{2_1\to 0_1}$	$\frac{2_\beta\to 0_\beta}{2_1\to 0_1}$	σ
^{100}Mo	Exp	1.86(11)	2.54(38)	3.32(49)		2.49(12)	0	0.97(49)	0.38(11)	
	H	1.88	3.33	6.16	11.26	1.52	0.14	2.29	2.98	0.84
	C	2.25	1.41	0.76	0.43	1.41	4.88	0.73	0.07	1.06
^{108}Ru	Exp	1.65(20)								
	H	1.59	2.13	2.88	3.98	0.56	0.08	0.58	2.04	0.05
	C	2.61	1.71	0.93	0.53	1.98	5.82	0.77	0.04	0.96
^{128}Xe	Exp	1.47(15)	1.94(20)	2.39(30)						
	H	1.73	2.65	4.22	6.83	0.97	0.11	1.25	2.52	0.48
	C	2.44	1.57	0.85	0.48	1.71	5.39	0.75	0.05	0.83
^{146}Nd	Exp	1.47(39)								
	H	1.80	2.97	5.11	8.84	1.23	0.13	1.73	2.76	0.33
	C	2.35	1.50	0.81	0.45	1.57	5.15	0.74	0.06	0.88
^{148}Nd	Exp	1.62	1.76	1.69		0.54	0.25	0.28		
	H	1.71	2.57	3.99	6.33	0.90	0.11	1.14	2.45	0.59
	C	2.49	1.61	0.87	0.49	1.78	5.50	0.76	0.05	0.91
^{150}Sm	Exp	1.93(30)	2.63(88)	2.98(158)		0.93(9)			1.93	
	H	1.78	2.86	4.81	8.15	1.14	0.12	1.56	2.68	0.57
	C	2.40	1.53	0.83	0.47	1.64	5.26	0.74	0.05	0.76
^{172}Os	Exp	1.56(6)	1.82(10)	1.99(11)	2.29(26)	0.33(5)	0.04	0.12(1)	0.62(6)	
	H	1.70	2.52	3.87	6.06	0.87	0.11	1.07	2.42	1.01
	C	2.50	1.62	0.88	0.50	1.81	5.54	0.76	0.05	1.16
^{190}Hg	Exp									
	H	1.77	2.83	4.73	7.97	1.12	0.12	1.52	2.66	
	C	2.37	1.51	0.82	0.46	1.60	5.20	0.74	0.062	
^{192}Pt	Exp	1.56	1.22							
	H	1.68	2.46	3.72	5.75	0.82	0.10	1.00	2.37	0.48
	C	2.50	1.62	0.88	0.49	1.80	5.54	0.76	0.050	0.09
^{196}Pt	Exp	1.48(2)	1.80(10)	1.92(25)				=0	0.12	
	H	1.70	2.54	3.93	6.19	0.89	0.11	1.10	2.43	0.60
	C	2.48	1.60	0.87	0.49	1.77	5.48	0.759	0.05	0.63

fairly better than those obtained with Coulomb one. The best candidate nuclei for the model with Hulthén potential are: ^{172}Os, ^{192}Pt, ^{196}Pt and ^{190}Hg.

7 Conclusion

In this work, we have solved the Bohr-Mottelson Hamiltonian in the γ-rigid regime within the minimal length formalism with two well-known potentials: Coulomb and Hulthén.

From the comparison between the energy spectra and transition probabilities in the two cases: presence and absence of the minimal length, one can conclude that the obtained results with Hulthén potential within the ML are better. This latter reproduces well the $X(3)$ candidates which already have been obtained including the predicted new one: ^{190}Hg.

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