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Wigner Function Moments Method and RPA – Similarity and Difference

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Abstract. Two methods describing the nuclear collective motion, the Wigner Function Moments (WFM) and the Random Phase Approximation (RPA), are compared on the example of the nuclear scissors mode description.

1 Introduction

The aim of this paper is the systematic comparison of two methods to describe the collective motion: Wigner Function Moments (WFM) and Random Phase Approximation (RPA).

RPA is very well known, but it is necessary to say several words about WFM method. The idea of the WFM method is based on the virial theorems of Chandrasekhar and Lebovitz. These theorems were derived by the authors in fifties in a series of papers, the results of which were summarized in the book [1]. The old astrophysical problems were considered: figures of equilibrium of rotating self–gravitating masses (planets and stars) and their vibration eigenfrequencies. In the classical mechanics the dynamics of such objects is described with the help of the well–known equations of hydrodynamics, the Euler equation and the continuity equation, which usually lead to very complicated mathematical problems. Chandrasekhar and Lebovitz have shown that the solution of these problems can be found in an essentially simpler and elegant way if one works with moments of the Euler equation (virial theorems).

Following to this idea one writes the dynamical equations for various multipole phase space moments of a nucleus (WFM), instead of writing the equations of motion for microscopic amplitudes of particle hole excitations (RPA). Let us consider, as an example, the competition of two methods in the description of the nuclear scissors mode.

2 WFM

The basis of the method is the Time Dependent Hartree-Fock (TDHF) equation for the one-body density matrix $\rho^\tau(\mathbf{r}_1, \mathbf{r}_2, t) = \langle \mathbf{r}_1, t|\hat{\rho}^\tau|\mathbf{r}_2, t\rangle$:

$$i\hbar \frac{\partial \hat{\rho}^\tau}{\partial t} = \left[\hat{h}^\tau, \hat{\rho}^\tau\right], \tag{1}$$

where $\hat{h}^{\tau}$ is the one-body self-consistent mean field Hamiltonian and τ is an isotopic index. With the help of Fourier (Wigner) transformation

$$f^{\tau}(\mathbf{r},\mathbf{p},t) = \int d^3s\ \exp(-i\mathbf{p}\cdot\mathbf{s}/\hbar)\rho^{\tau}(\mathbf{r}+\frac{\mathbf{s}}{2},\mathbf{r}-\frac{\mathbf{s}}{2},t) \tag{2}$$

the density matrix $\rho^{\tau}(\mathbf{r}_1,\mathbf{r}_2,t)$ is transformed into the Wigner function $f^{\tau}(\mathbf{r},\mathbf{p},t)$ and TDHF equation for a density matrix is transformed into TDHF equation for a Wigner function. Integrating this equation over a phase space with the weights

$$\{r\otimes r\}_{\lambda\mu},\quad \{p\otimes p\}_{\lambda\mu},\quad \{r\otimes p\}_{\lambda\mu},$$

where $\{r\otimes p\}_{\lambda\mu} = \sum_{\sigma,\nu} C^{\lambda\mu}_{1\sigma,1\nu} r_{\sigma} p_{\nu}$ is a tensor product, one derives dynamical equations for the following second order moments, collective variables of WFM method:

$$\begin{aligned}
R^{\tau}_{\lambda\mu}(t) &= 2(2\pi\hbar)^{-3}\int d\mathbf{p}\int d\mathbf{r}\,\{r\otimes r\}_{\lambda\mu} f^{\tau}(\mathbf{r},\mathbf{p},t),\\
P^{\tau}_{\lambda\mu}(t) &= 2(2\pi\hbar)^{-3}\int d\mathbf{p}\int d\mathbf{r}\,\{p\otimes p\}_{\lambda\mu} f^{\tau}(\mathbf{r},\mathbf{p},t),\\
L^{\tau}_{\lambda\mu}(t) &= 2(2\pi\hbar)^{-3}\int d\mathbf{p}\int d\mathbf{r}\,\{r\otimes p\}_{\lambda\mu} f^{\tau}(\mathbf{r},\mathbf{p},t).
\end{aligned} \tag{3}$$

Let us use the simple Hamiltonian:

$$\begin{aligned}
H = &\sum_{i=1}^{A}\left(\frac{\hat{\mathbf{p}}_i^2}{2m} + \frac{1}{2}m\omega^2\mathbf{r}_i^2\right) + \bar{\kappa}\sum_{\mu=-2}^{2}(-1)^{\mu}\sum_{i}^{Z}\sum_{j}^{N} q_{2\mu}(\mathbf{r}_i)q_{2-\mu}(\mathbf{r}_j)\\
&+\frac{1}{2}\kappa\sum_{\mu=-2}^{2}(-1)^{\mu}\left\{\sum_{i\neq j}^{Z} q_{2\mu}(\mathbf{r}_i)q_{2-\mu}(\mathbf{r}_j) + \sum_{i\neq j}^{N} q_{2\mu}(\mathbf{r}_i)q_{2-\mu}(\mathbf{r}_j)\right\},
\end{aligned} \tag{4}$$

where $q_{2\mu} = \sqrt{16\pi/5}\, r^2 Y_{2\mu}$ and N, Z are the number of neutrons and protons respectively.

The integration yields the set of coupled nonlinear dynamical equations:

$$\begin{aligned}
&\frac{d}{dt}R^{\tau}_{\lambda\mu} - \frac{2}{m}L^{\tau}_{\lambda\mu} = 0,\\
&\frac{d}{dt}L^{\tau}_{\lambda\mu} - \frac{1}{m}P^{\tau}_{\lambda\mu} + m\,\omega^2 R^{\tau}_{\lambda\mu} - 12\sqrt{5}\sum_{j=0,2}\sqrt{2j+1}\left\{\begin{smallmatrix}1&1&j\\2&\lambda&1\end{smallmatrix}\right\}\{Z^{\tau}_2\otimes R^{\tau}_j\}_{\lambda\mu} = 0,\\
&\frac{d}{dt}P^{\tau}_{\lambda\mu} + 2m\,\omega^2 L^{\tau}_{\lambda\mu} - 24\sqrt{5}\sum_{j=0,1,2}\sqrt{2j+1}\left\{\begin{smallmatrix}1&1&j\\2&\lambda&1\end{smallmatrix}\right\}\{Z^{\tau}_2\otimes L^{\tau}_j\}_{\lambda\mu} = 0.
\end{aligned}$$

Here $\left\{\begin{smallmatrix}1&1&j\\2&\lambda&1\end{smallmatrix}\right\}$ is the Wigner $6j$-symbol, $\lambda = 0,1,2,\ R_{1\mu} = P_{1\mu} \equiv 0$ and

$$Z^{\mathrm{n}}_{2\mu} = \kappa R^{\mathrm{n}}_{2\mu} + \bar{\kappa}R^{\mathrm{p}}_{2\mu}\,,\qquad Z^{\mathrm{p}}_{2\mu} = \kappa R^{\mathrm{p}}_{2\mu} + \bar{\kappa}R^{\mathrm{n}}_{2\mu}\,.$$

These equations can be solved analytically in a small amplitude approximation. To this end all variables are represented as the sum of the equilibrium value plus infinitesimally small variation $X_{\lambda\mu}(t) = X^{\rm eq}_{\lambda\mu} + \mathcal{X}_{\lambda\mu}(t)$, and one neglects by squares of $\mathcal{X}_{\lambda\mu}(t)$. It is convenient to rewrite equations in terms of isoscalar and isovector variables:

$$X_{\lambda\mu}(t) = X^{\rm n}_{\lambda\mu}(t)+X^{\rm p}_{\lambda\mu}(t), \quad \bar{X}_{\lambda\mu}(t) = X^{\rm n}_{\lambda\mu}(t)-X^{\rm p}_{\lambda\mu}(t), \quad X \equiv \{R, L, P\}.$$

Solving characteristic equation of the isovector set of equations one finds analytic expressions for energies of IVGQR and a scissors mode:

$$\begin{aligned} E^2_{\rm IVGQR} &= (2\hbar\bar{\omega})^2\left(1+\frac{\delta}{3}+\sqrt{(1+\frac{\delta}{3})^2-\frac{3}{4}\delta^2}\right), \\ E^2_{\rm scis} &= (2\hbar\bar{\omega})^2\left(1+\frac{\delta}{3}-\sqrt{(1+\frac{\delta}{3})^2-\frac{3}{4}\delta^2}\right), \end{aligned} \tag{5}$$

where

$$\bar{\omega}^2 = \omega_0^2(1+\frac{4}{3}\delta)^{-2/3}(1-\frac{2}{3}\delta)^{-1/3}.$$

The reduced transition probabilities are calculated with the help of the theory of a linear response of a system to a weak external field

$$\hat{F}(t) = \hat{F}\exp(-i\Omega t) + \hat{F}^{\dagger}\exp(i\Omega t), \tag{6}$$

where $\hat{F} = \sum_{s=1}^{A}\hat{f}_s$ is a one-body operator. A convenient form of the response theory is e.g. given by Lane [2]. The matrix elements of the operator $\hat{F}$ obey the relation

$$|\langle\nu|\hat{F}|0\rangle|^2 = \hbar\lim_{\Omega\to\Omega_\nu}(\Omega-\Omega_\nu)\overline{\langle\psi|\hat{F}|\psi\rangle\exp(-i\Omega t)}, \tag{7}$$

where $|0\rangle$ and $|\nu\rangle$ are the stationary wave functions of unperturbed ground and excited states; ψ is the wave function of the perturbed ground state. To calculate the magnetic transition probability, it is necessary to excite the system with the following external field:

$$\hat{F} = \sum_{s=1}^{Z}\hat{f}_{\lambda\mu'}(s), \qquad \hat{f}_{\lambda\mu'} = -i\frac{2}{\lambda+1}\nabla(r^{\lambda}Y_{\lambda\mu'})\cdot[\mathbf{r}\times\nabla]\mu_N,$$

where $\mu_N = e\hbar/2mc$. One finds for $\mu' = 1$:

$$B(M1)_\nu = \frac{3}{8\pi}m\bar{\omega}^2 Q_{00}\delta^2\frac{E_\nu^2-2(1+\delta/3)\hbar^2\bar{\omega}^2}{E_\nu[E_\nu^2-(2\hbar\bar{\omega})^2(1+\delta/3)]}\,\mu_N^2, \tag{8}$$

where $Q_{00} = A\langle r^2\rangle$.

3 RPA

RPA equations in the notation of [3] are

$$\begin{aligned}
&\sum_{n,j}\left\{\left[\delta_{ij}\delta_{mn}(\epsilon_m-\epsilon_i)+\bar{v}_{mjin}\right]X_{nj}+\bar{v}_{mnij}Y_{nj}\right\}=\hbar\Omega X_{mi},\\
&\sum_{n,j}\left\{\bar{v}_{ijmn}X_{nj}+\left[\delta_{ij}\delta_{mn}(\epsilon_m-\epsilon_i)+\bar{v}_{inmj}\right]Y_{nj}\right\}=-\hbar\Omega Y_{mi}.
\end{aligned} \tag{9}$$

where X_{mi} and Y_{mi} are the coefficients of the collective particle-hole operator

$$O^\dagger_\nu=\sum_{mi}X^\nu_{mi}a^\dagger_m a_i-\sum_{mi}Y^\nu_{mi}a^\dagger_i a_m\,,\qquad |\nu\rangle=O^\dagger_\nu|0\rangle.$$

According to the definition of the schematic model matrix elements of the residual interaction corresponding to the Hamiltonian (4) are written as

$$\bar{v}_{mjin}=\kappa_{\tau\tau'}\mathcal{Q}^{\tau*}_{im}\mathcal{Q}^{\tau'}_{jn}$$

with $\mathcal{Q}_{im}\equiv<i|q_{21}|m>$ and $\kappa_{\rm nn}=\kappa_{\rm pp}=\kappa,\quad \kappa_{\rm np}=\bar{\kappa}$. This interaction distinguishes between protons and neutrons, so we have to introduce the isospin indices τ, τ' into the set of RPA equations (9):

$$\begin{aligned}
(\epsilon^\tau_m-\epsilon^\tau_i)X^\tau_{mi}&+\sum_{n,j,\tau'}\kappa_{\tau\tau'}\mathcal{Q}^{\tau*}_{im}\mathcal{Q}^{\tau'}_{jn}X^{\tau'}_{nj}\\
&+\sum_{n,j,\tau'}\kappa_{\tau\tau'}\mathcal{Q}^{\tau*}_{im}\mathcal{Q}^{\tau'}_{nj}Y^{\tau'}_{nj}=\hbar\Omega X^\tau_{mi},\\
\sum_{n,j,\tau'}\kappa_{\tau\tau'}\mathcal{Q}^{\tau*}_{mi}\mathcal{Q}^{\tau'}_{jn}X^{\tau'}_{nj}&+(\epsilon^\tau_m-\epsilon^\tau_i)Y^\tau_{mi}\\
&+\sum_{n,j,\tau'}\kappa_{\tau\tau'}\mathcal{Q}^{\tau*}_{mi}\mathcal{Q}^{\tau'}_{nj}Y^{\tau'}_{nj}=-\hbar\Omega Y^\tau_{mi}.
\end{aligned} \tag{10}$$

The solution of these equations is

$$X^\tau_{mi}=\frac{\mathcal{Q}^{\tau*}_{im}}{E-\epsilon^\tau_{mi}}K^\tau,\quad Y^\tau_{mi}=-\frac{\mathcal{Q}^{\tau*}_{mi}}{E+\epsilon^\tau_{mi}}K^\tau \tag{11}$$

with $E=\hbar\Omega$, $\epsilon^\tau_{mi}=\epsilon^\tau_m-\epsilon^\tau_i$ and $K^\tau=\sum_{\tau'}\kappa_{\tau\tau'}C^{\tau'}$.

The constant C^τ is defined as $C^\tau=\sum_{n,j}(\mathcal{Q}^\tau_{jn}X^\tau_{nj}+\mathcal{Q}^\tau_{nj}Y^\tau_{nj})$. Using here the expressions for X^τ_{nj} and Y^τ_{nj} given above, one derives the useful relation

$$C^\tau=2S^\tau K^\tau=2S^\tau\sum_{\tau'}\kappa_{\tau\tau'}C^{\tau'}, \tag{12}$$

where the following notation is introduced:

$$S^\tau=\sum_{mi}|\mathcal{Q}^\tau_{mi}|^2\frac{\epsilon^\tau_{mi}}{E^2-(\epsilon^\tau_{mi})^2}. \tag{13}$$

Let us write out the relation (12) in detail

$$\begin{aligned} C^{\rm n} - 2S^{\rm n}(\kappa C^{\rm n} + \bar\kappa C^{\rm p}) = 0,\\ C^{\rm p} - 2S^{\rm p}(\bar\kappa C^{\rm n} + \kappa C^{\rm p}) = 0. \end{aligned} \tag{14}$$

The condition for an existence of a nontrivial solution of this set of equations gives the secular equation

$$(1 - 2S^{\rm n}\kappa)(1 - 2S^{\rm p}\kappa) - 4S^{\rm n}S^{\rm p}\bar\kappa^2 = 0. \tag{15}$$

The detailed expression for the isovector secular equation is

$$\frac{1}{2\kappa_1} = \sum_{mi} |\mathcal{Q}_{mi}|^2 \frac{\epsilon_{mi}}{E^2 - \epsilon_{mi}^2}. \tag{16}$$

The operator $\mathcal{Q} = q_{21}$ has only two types of nonzero matrix elements $\mathcal{Q}_{mi}$ in the deformed oscillator basis. Matrix elements of the first type couple states of the same major shell. All corresponding transition energies are degenerate: $\epsilon_m - \epsilon_i = \hbar(\omega_x - \omega_z) \equiv \epsilon_0$. Matrix elements of the second type couple states of the different major shells with $\Delta N = 2$. All corresponding transition energies are degenerate too: $\epsilon_m - \epsilon_i = \hbar(\omega_x + \omega_z) \equiv \epsilon_2$. Therefore, the secular equation can be rewritten as

$$\frac{1}{2\kappa_1} = \frac{\epsilon_0 \mathcal{Q}_0}{E^2 - \epsilon_0^2} + \frac{\epsilon_2 \mathcal{Q}_2}{E^2 - \epsilon_2^2}. \tag{17}$$

The sums $\mathcal{Q}_0 = \sum\limits_{mi(\Delta N=0)} |\mathcal{Q}_{mi}|^2$ and $\mathcal{Q}_2 = \sum\limits_{mi(\Delta N=2)} |\mathcal{Q}_{mi}|^2$ can be calculated analytically:

$$\mathcal{Q}_0 = \frac{Q_{00}}{m\bar\omega^2}\epsilon_0, \quad \mathcal{Q}_2 = \frac{Q_{00}}{m\bar\omega^2}\epsilon_2. \tag{18}$$

As a result one gets analytic expressions for energies, which coincide with the ones derived by WFM method (5).

The transition probability for the one-body operator $\hat F = \sum\limits_{s=1}^{A} \hat f_s$ is calculated by means of the formulae

$$\langle 0|\hat F^\tau|\nu\rangle = \sum_{mi}(f^\tau_{im}X^{\tau\nu}_{mi} + f^\tau_{mi}Y^{\tau\nu}_{mi}), \quad \langle \nu|\hat F^\tau|0\rangle = \sum_{mi}(f^\tau_{mi}X^{\tau\nu}_{mi} + f^\tau_{im}Y^{\tau\nu}_{mi}).$$

It is easy to derive analytic expressions, which coincide with that ones derived by WFM method (8).

So, one can conclude that WFM method and RPA are identical, is not it?! Not at all. They are **similar**, but not **identical**! Their results coincide in the case of exactly soluble models, as it happened in the considered example, and not more!

4 Similarities and differences

4.1 Similarities

1) The basis of both methods is the same: Time Dependent Hartree–Fock (TDHF).
2) The small amplitude approximation.
3) 1p-1h excitations: evident in RPA and implicit in WFM, because the idempotent property $\rho^2 = \rho$ of the density matrix must be fulfilled.

4.2 Differences

1) Strictly speaking the small amplitude approximation is not compulsory in the WFM method – it allows one to study the **large amplitude motion** too. For example multiphonon excitations of giant resonances were described in [4]. So, from this point of view the WFM method is more general than the RPA.
2) The principal approximations of two methods are absolutely different.

a) In RPA it is the quasi-boson approximation,

$$\langle RPA|[a_i^\dagger a_m, a_n^\dagger a_j]|RPA\rangle = \delta_{ij}\delta_{mn} - \delta_{mn}\langle RPA|a_j a_i^\dagger|RPA\rangle \\ - \delta_{ij}\langle RPA|a_n^\dagger a_m|RPA\rangle \simeq \langle HF|[a_i^\dagger a_m, a_n^\dagger a_j]|HF\rangle = \delta_{ij}\delta_{mn},$$

the quality of which can be checked only from realistic calculations.

b) In WFM one neglects by the influence of higher multipolarity moments on the dynamics of lower multipolarity moments. This approximation seems quite natural on the intuitive level and is confirmed by calculations [5].

For example, in the case with spin degrees of freedom one gets the following set of dynamic equations for second order moments (which have now spin indexes $\uparrow\downarrow$, $\downarrow\uparrow$, $\uparrow\uparrow$, $\downarrow\downarrow$, $X^\pm = X^{\uparrow\uparrow} \pm X^{\downarrow\downarrow}$). Some of these equations contain integral terms, who generate fourth order moments. Namely they are neglected here.

$$\dot{L}^+_{\lambda\mu} = \frac{1}{m}P^+_{\lambda\mu} - m\,\omega^2 R^+_{\lambda\mu} + 2\sqrt{5}\sum_{j=0}^{2}\sqrt{2j+1}\{^{11j}_{2\lambda 1}\}\{Z_2^+ \otimes R_j^+\}_{\lambda\mu} \\ - i\hbar\frac{\eta}{2}[\mu L^-_{\lambda\mu} + \lambda^-_\mu L^{\uparrow\downarrow}_{\lambda\mu+1} + \lambda^+_\mu L^{\downarrow\uparrow}_{\lambda\mu-1}],$$

$$\dot{L}^-_{\lambda\mu} = \frac{1}{m}P^-_{\lambda\mu} - m\,\omega^2 R^-_{\lambda\mu} + 2\sqrt{5}\sum_{j=0}^{2}\sqrt{2j+1}\{^{11j}_{2\lambda 1}\}\{Z_2^+ \otimes R_j^-\}_{\lambda\mu} \\ + i\eta\sqrt{2}\int d(\mathbf{p},\mathbf{r})\{r\otimes p\}_{\lambda\mu}[l_1 f^{\uparrow\downarrow} + l_{-1} f^{\downarrow\uparrow}] \\ - i\hbar\frac{\eta}{2}\mu L^+_{\lambda\mu} - \frac{\hbar^2}{2}\eta\delta_{\lambda,1}[\delta_{\mu,-1}F^{\uparrow\downarrow} + \delta_{\mu,1}F^{\downarrow\uparrow}],$$

$$
\begin{aligned}
\dot{L}^{\uparrow\downarrow}_{\lambda\mu+1} =& \frac{1}{m}P^{\uparrow\downarrow}_{\lambda\mu+1} - m\,\omega^2 R^{\uparrow\downarrow}_{\lambda\mu+1} + 2\sqrt{5}\sum_{j=0}^{2}\sqrt{2j+1}\{^{11j}_{2\lambda 1}\}\{Z_2^+\otimes R_j^{\uparrow\downarrow}\}_{\lambda\mu+1}\\
&- i\frac{\eta}{\sqrt{2}}\int d(\mathbf{p},\mathbf{r})\{r\otimes p\}_{\lambda\mu+1}[l_{-1}f^- - \sqrt{2}l_0 f^{\uparrow\downarrow}]\\
&- i\hbar\frac{\eta}{4}\lambda_\mu^- L^+_{\lambda\mu} + \frac{\hbar^2}{2}\eta\delta_{\lambda,1}[\delta_{\mu,0}F^- + \frac{1}{\sqrt{2}}\delta_{\mu,-1}F^{\uparrow\downarrow}],\\
\dot{L}^{\downarrow\uparrow}_{\lambda\mu-1} =& \frac{1}{m}P^{\downarrow\uparrow}_{\lambda\mu-1} - m\,\omega^2 R^{\downarrow\uparrow}_{\lambda\mu-1} + 2\sqrt{5}\sum_{j=0}^{2}\sqrt{2j+1}\{^{11j}_{2\lambda 1}\}\{Z_2^+\otimes R_j^{\downarrow\uparrow}\}_{\lambda\mu-1}\\
&- i\frac{\eta}{\sqrt{2}}\int d(\mathbf{p},\mathbf{r})\{r\otimes p\}_{\lambda\mu-1}[l_1 f^- + \sqrt{2}l_0 f^{\downarrow\uparrow}]\\
&- i\hbar\frac{\eta}{4}\lambda_\mu^+ L^+_{\lambda\mu} + \frac{\hbar^2}{4}\eta\delta_{\lambda,1}[\delta_{\mu,0}F^- - \sqrt{2}\delta_{\mu,1}F^{\downarrow\uparrow}],\\
\dot{F}^- =& 2\eta[L^{\downarrow\uparrow}_{1-1} + L^{\uparrow\downarrow}_{11}],\\
\dot{F}^{\uparrow\downarrow} =& -\eta[L^-_{1-1} - \sqrt{2}L^{\uparrow\downarrow}_{10}],\\
\dot{F}^{\downarrow\uparrow} =& -\eta[L^-_{11} + \sqrt{2}L^{\downarrow\uparrow}_{10}],\\
\dot{R}^+_{\lambda\mu} =& \frac{2}{m}L^+_{\lambda\mu}\\
&- i\hbar\frac{\eta}{2}[\mu R^-_{\lambda\mu} + \lambda_\mu^- R^{\uparrow\downarrow}_{\lambda\mu+1} + \lambda_\mu^+ R^{\downarrow\uparrow}_{\lambda\mu-1}],\\
\dot{R}^-_{\lambda\mu} =& \frac{2}{m}L^-_{\lambda\mu} - i\hbar\frac{\eta}{2}\mu R^+_{\lambda\mu} + i\eta\sqrt{2}\int d(\mathbf{p},\mathbf{r})\{r\otimes r\}_{\lambda\mu}[l_1 f^{\uparrow\downarrow} + l_{-1}f^{\downarrow\uparrow}],\\
\dot{R}^{\uparrow\downarrow}_{\lambda\mu+1} =& \frac{2}{m}L^{\uparrow\downarrow}_{\lambda\mu+1} - i\frac{\eta}{\sqrt{2}}\int d(\mathbf{p},\mathbf{r})\{r\otimes r\}_{\lambda\mu+1}[l_{-1}f^- - \sqrt{2}l_0 f^{\uparrow\downarrow}]\\
&- i\hbar\frac{\eta}{4}\lambda_\mu^- R^+_{\lambda\mu},\\
\dot{R}^{\downarrow\uparrow}_{\lambda\mu-1} =& \frac{2}{m}L^{\downarrow\uparrow}_{\lambda\mu-1} - i\frac{\eta}{\sqrt{2}}\int d(\mathbf{p},\mathbf{r})\{r\otimes r\}_{\lambda\mu-1}[l_1 f^- + \sqrt{2}l_0 f^{\downarrow\uparrow}]\\
&- i\hbar\frac{\eta}{4}\lambda_\mu^+ R^+_{\lambda\mu},\\
\dot{P}^+_{\lambda\mu} =& -2m\,\omega^2 L^+_{\lambda\mu} + 4\sqrt{5}\sum_{j=0}^{2}\sqrt{2j+1}\{^{11j}_{2\lambda 1}\}\{Z_2^+\otimes L_j^+\}_{\lambda\mu}\\
&- i\hbar\frac{\eta}{2}[\mu P^-_{\lambda\mu} + \lambda_\mu^- P^{\uparrow\downarrow}_{\lambda\mu+1} + \lambda_\mu^+ P^{\downarrow\uparrow}_{\lambda\mu-1}],\\
\dot{P}^-_{\lambda\mu} =& -2m\,\omega^2 L^-_{\lambda\mu} + 4\sqrt{5}\sum_{j=0}^{2}\sqrt{2j+1}\{^{11j}_{2\lambda 1}\}\{Z_2^+\otimes L_j^-\}_{\lambda\mu}\\
&- i\hbar\frac{\eta}{2}\mu P^+_{\lambda\mu} + i\eta\sqrt{2}\int d(\mathbf{p},\mathbf{r})\{p\otimes p\}_{\lambda\mu}[l_1 f^{\uparrow\downarrow} + l_{-1}f^{\downarrow\uparrow}],
\end{aligned}
$$

$$\dot{P}^{\uparrow\downarrow}_{\lambda\mu+1} = -2m\,\omega^2 L^{\uparrow\downarrow}_{\lambda\mu+1} + 4\sqrt{5}\sum_{j=0}^{2}\sqrt{2j+1}\{^{11j}_{2\lambda1}\}\{Z_2^+\otimes L_j^{\uparrow\downarrow}\}_{\lambda\mu+1}$$
$$- i\hbar\frac{\eta}{4}\lambda^-_\mu P^+_{\lambda\mu} - i\frac{\eta}{\sqrt{2}}\int d(\mathbf{p},\mathbf{r})\{p\otimes p\}_{\lambda\mu+1}[l_{-1}f^- - \sqrt{2}l_0 f^{\uparrow\downarrow}],$$
$$\dot{P}^{\downarrow\uparrow}_{\lambda\mu-1} = -2m\,\omega^2 L^{\downarrow\uparrow}_{\lambda\mu-1} + 4\sqrt{5}\sum_{j=0}^{2}\sqrt{2j+1}\{^{11j}_{2\lambda1}\}\{Z_2^+\otimes L_j^{\downarrow\uparrow}\}_{\lambda\mu-1}$$
$$- i\hbar\frac{\eta}{4}\lambda^+_\mu P^+_{\lambda\mu} - i\frac{\eta}{\sqrt{2}}\int d(\mathbf{p},\mathbf{r})\{p\otimes p\}_{\lambda\mu-1}[l_1 f^- + \sqrt{2}l_0 f^{\downarrow\uparrow}],$$

where $\lambda^\pm_\mu = \sqrt{(\lambda\pm\mu)(\lambda\mp\mu+1)}$.

3. Being formulated in terms of collective variables, WFM method does not meet the problem of basis, contrary to RPA, which is formulated in terms of creation and annihilation operators.

4. Multiplying the TDHF equation by some functions ($\{r\otimes r\}_{\lambda\mu}$, $\{p\otimes p\}_{\lambda\mu}$, $\{r\otimes p\}_{\lambda\mu}$, in our case) one does not destroy its symmetries. As a consequence, all conservation laws are fulfilled (for example, energy and angular momentum) and spurious states don't appear.

5. WFM method allows one more direct physical interpretation of the studied phenomenon because every collective variable has the clear physical sense:

$Q_{2\mu}$ — quadrupole moment,

$P_{2\mu}$ — quadrupole moment in a momentum space (Fermi surface deformation),

$L_{1\mu}$ — angular momentum (rotation).

For example, from the RPA dynamic equations it follows that low lying 1^+ excitations are just transitions inside of one shell, and nothing more.

From WFM dynamic equations it follows that low lying 1^+ excitations are generated by all three above mentioned variables – this means that the relative rotation of protons and neutrons ($\bar{L}_{11}$ variable) is accompanied by the Fermi surface deformation ($\bar{P}_{21}$ variable) and by the isovector quadrupole deformation i.e. IVGQR ($\bar{Q}_{21}$ variable).

WFM method allows one to distinguish three types of scissors modes in the case with spin degrees of freedom, whereas it is impossible in the frame of RPA. Actually, the existence of three scissors states is naturally explained by combinatorics – there are only three ways to divide the four different kinds of objects (spin up and spin down, protons and neutrons in our case) into two pairs:

i) spin-up and spin-down protons oscillate versus the corresponding neutrons (the conventional scissors mode), the responsible variable is

$$\bar{L}^+_{11} = (L^{\uparrow\uparrow}_{11} + L^{\downarrow\downarrow}_{11})^{\rm n} - (L^{\uparrow\uparrow}_{11} + L^{\downarrow\downarrow}_{11})^{\rm p},$$

ii) protons and neutrons, both spin-up, oscillate versus same with spin-down, the responsible variable is $L^-_{11} = (L^{\rm n}_{11} + L^{\rm p}_{11})^{\uparrow\uparrow} - (L^{\rm n}_{11} + L^{\rm p}_{11})^{\downarrow\downarrow}$,

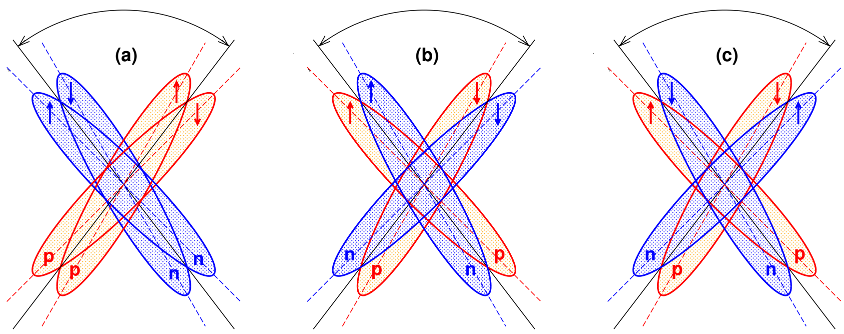

Figure 1. Schematic representation of three scissors modes: (a) spin-scalar isovector (conventional, orbital scissors), (b) spin-vector isoscalar (spin scissors), (c) spin-vector isovector (spin scissors). Arrows show the direction of spin projections; p – protons, n – neutrons. The small angle spread between the various distributions is only for presentation purposes.

iii) protons spin-up with neutrons spin-down oscillate versus protons spin-down with neutrons spin-up, the responsible variable is

$$\bar{L}^{-}_{11} = (L^{\uparrow\uparrow\mathrm{n}}_{11} + L^{\downarrow\downarrow\mathrm{p}}_{11}) - (L^{\uparrow\uparrow\mathrm{p}}_{11} + L^{\downarrow\downarrow\mathrm{n}}_{11}).$$

5 Conclusion

WFM and RPA are complementary methods: first one describes averaged, summarized characteristics of the phenomenon, whereas second one allows one to study its fine structure.

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