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Conformal Symmetry, Unitary Limit and Collective Nuclear States

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Abstract. This contribution discusses the exploration of conformal symmetry in intermediate states of compound nuclei with the aid of the unitary limit. The latter manifests itself in a heavy A+2n compound nucleus in view of the tuning of the scattering length between two cold neutrons (2n) and the target nucleus (A). Fluctuations of cross sections are identified as the experimental means through which the unitary limit can be examined in nuclear physics.

1 Introduction

Low-lying, collective nuclear states in heavy, even-even nuclei constitute the primary example of how simple regularity patterns emerge from the complex collective motions of bound nucleons. The first model that predicted such remarkable regularities in nuclear spectra and transition rates was the Bohr Hamiltonian [1]. It established the concepts of nuclear shape and deformation as the classical analogs of the rotational spectra of atomic nuclei. Then, with the establishment of nuclear shell effects, the role of particle-hole excitations illustrated the emergence of collective effects out of single particle motions [2].

The first samples of analyzing collective motions of nucleons in terms of interacting bosons appeared in the late sixties in Iachello's Ph.D. thesis [3]. However, during the previous decade, intensive research led by Hermann Feshbach [4] on the formalism of intermediate states of compound nuclei had already revealed the tremendous complexity of nuclear excitations in the continuum. Intermediate states of $A + 1n$ compound nuclei - the so-called doorway states - exemplified one aspect of that complexity. In the 70s, doorway states were experimentally investigated via the fluctuations of the cross-sections as bound states in the continuum. Today these states are included in the formalism of the continuum Shell-Model [5].

The establishment of the Interacting Boson Model [6] produced a simple and operational classification scheme of the symmetries exhibited in bound, low-lying collective states of heavy, even-even atomic nuclei under the $U(6)$ symmetry group. The three dynamical symmetry limits of the model $U(5)$, $O(6)$, and $SU(3)$ arise under three different subgroup chains of the $U(6)$. A spin zero s boson and a spin two $\mathbf{d}$ boson are its building blocks. The s and $\mathbf{d}$ bosons are identified as valence nucleon pairs of total angular momentum zero and two, respectively. An overall fixed boson number N_b determines an atomic nucleus. A Schrödinger equation for the $O(6)$ limit of the IBM has been developed by the Mexico school [7] where one focuses on the boson number radius ρ. That radius remains invariant under the $O(6)$ group of rotations for the six-dimensional harmonic oscillator of IBM's $U(6)$ symmetry group.

On the other hand, cold and dilute atomic gases manifest the so-called unitary limit in the vicinity of Feshbach resonances [9]. The unitary limit is a theoretical benchmark that accompanies interesting physical phenomena like the BCS-Bose Einstein Condensation crossover and a quantum critical point. In parallel, it refers to a strong coupling problem described by a Conformal Field Theory [10]. The theoretical application of the unitary limit in nuclear physics was first realized in light nuclei [11]. It emerged parallel with the development of Effective Field Theories [12]. Conformal invariance arises at the critical point of a second-order phase transition [13] as well as in Quantum Chromo Dynamics at the limit of a large number of gluons [14]. In that perspective, introducing the unitary limit in heavy even-even nuclei commences the investigation of algebraic relations between the symmetries of the IBM with the conformal symmetry of second-order critical points. In parallel, it explores algebraic relations with the limit of conformal invariance in a strong coupling problem (unitary limit) that is amenable to be incorporated afterward with QCD.

Notwithstanding the profound complexity of the fluctuations of cross sections in $A + 1n$ compound nuclei [5], those fluctuations of intermediate states of $A + 2n$ compound nuclei are largely unexplored. This contribution points to the emergence of regularity patterns in fluctuations of cross sections of $A + 2n$ compound nuclei due to the representations of conformal symmetry. These representations result from the proposed examination of the unitary limit in nuclear physics via the energies and widths of those fluctuations.

2 Unitary Limit in Systems of Cold Atoms

One starts from the exhaustion of the unitarity bound in the cross-section of a scattering problem. In what follows, the discussion is restricted to s-wave scattering. The parameter that controls the deviation from the unitarity bound is the generalized scattering length $a(k)$ as defined by Bethe [16] through the effective range (r^*) expansion

$$k \cot \delta(k) = \frac{1}{a(k)} = \frac{1}{a} - \frac{1}{2} k^2 r^* + \cdots . \tag{1}$$

The scattering length a originates by the approximation of the phase shifts $\delta(k) \sim k(r - a)$ for $k \to 0$. It shows the effect of the scattering on the wavefunction by controlling its intercept with the horizontal axis of the radial distance r between the scattering particles. However, given the effective range r^* for the particles' interaction, it is the condition $1/a(k) = 0$ on the generalized scattering length that defines a resonance at the scattering amplitude [16]. At very low kinetic energies, $kr^* \ll 1$, that resonance condition implies an infinite value for the scattering length, i.e. $a \to \infty$. The latter amounts to maximize the interaction strength between two particles, $g = 4\pi a\hbar^2/m$ with m the mass of each particle, i.e., it reflects a strong coupling limit. The unitary limit refers to the infinite value of the scattering length a at low kinetic energy. A paradigm of its experimental observation has been achieved in the cold and dilute atomic gases [9].

The open-closed channel crossing during an atom-atom collision is the underlying mechanism that realizes the unitary limit. The open channel reflects a scattering state of two cold atoms, while the closed channel is the bound state of a diatomic molecule formed by these same cold atoms. Channels' crossing means the coincidence in the energy of two different channels. In the Feshbach formalism for reactions, the open-closed channel crossing is achieved through the resonating energy of the open channel with the energy of an intermediate state of the closed channel. That resonating energy gives rise to a resonance that manifests the intermediate state of the closed channel. In atomic and molecular physics, these resonances are the celebrated Feshbach resonances [17]. However, they were initially introduced in compound nuclei [4].

The intermediate state of the Feshbach formalism affects the wavefunction's scattering length a. In general, for low k, the element of the scattering matrix is expressed through the phase shifts as $S_0 = e^{2i\delta(k)} = e^{-2ika}$. In the presence of an intermediate state of energy E_m and width Γ_m, the quantity S_0 takes the form

$$S_0' = e^{2ika}\left(1 - i\frac{\Gamma_m}{E - E_m + i\Gamma_m/2}\right). \tag{2}$$

From the perspective of nuclear physics [4], Eq. (2) generates the scattering matrix element $S_0' = S_0 S_R$, with $S_R = 1 - i\Gamma_m/\left(E - E_m + i\Gamma_m/2\right)$ a fluctuating part that fluctuates rather rapidly with the energy by the resonating energies E_m and widths Γ_m. Now, like in atomic and molecular physics [17], one identifies the effect of that fluctuating part in the emergence of an effective scattering length $a_{\text{eff}} = a + a'$, with

$$\begin{aligned} e^{2ika'} &= 1 - i\frac{\Gamma_m}{E - E_m + i\frac{\Gamma_m}{2}}, \\ a_{\text{eff}} &= a + \frac{1}{2k}\tan^{-1}\left(\frac{\Gamma_m(E - E_m)}{(E - E_m)^2 + \frac{\Gamma_m^2}{4}}\right). \end{aligned} \tag{3}$$

The effective scattering length goes to infinity at the resonating energies $E = E_m$ of the open channel with the intermediate states. Therefore, a Feshbach resonance maximizes the scattering length. Experimentally in ultracold atoms, an external magnetic field tunes the energy of the channels to achieve their crossing that generates the Feshbach resonance.

3 $O(6)$ Limit of the IBM

The Schrödinger equation of the $O(6)$ limit of the IBM [7] reads

$$-\frac{\hbar^2}{2M}\left(\frac{1}{\rho^5}\frac{\partial}{\partial\rho}\rho^5\frac{\partial}{\partial\rho}-\frac{\sigma(\sigma+4)}{\rho^2}\right)\Phi(\rho)+\frac{1}{2}M\omega^2\rho^2\Phi(\rho)= \\ =\Big(N_b+\frac{6}{2}\Big)\hbar\omega\Phi(\rho). \quad (4)$$

It is realized in the six dimensional space ($d=6$) of the s and $\mathbf{d}$ bosons [7]. The boson number radius is $\rho=\sqrt{\beta^2+q_0^2}$. β is the quadrupole deformation of the nuclear surface defined by $\beta^2=\sum_{i=1}^{5}q_i^2$. The five quadrupole coordinates q_i define the five dimensional quadrupole plane where the $\mathbf{d}$ boson lives. The s boson coordinate q_0 is a sixth transversal coordinate to this plane. The numerator of the centrifugal term is the eigenvalues $\sigma(\sigma+4)$ of the angular wavefunctions that span the irreducible representations of the $O(6)$ group.

The radial solutions read

$$\Phi(\rho)=\frac{F_\sigma^J(\rho)}{\rho^{5/2}},\quad F_\sigma^J(\rho)=\frac{\rho^\sigma}{a_{ho}^\sigma}L_J^{\sigma+2}\Big(\frac{\rho^2}{a_{ho}^2}\Big)e^{-\rho^2/2a_{ho}^2}, \quad (5)$$

with eigenvalues

$$E(N_b)=\Big(\sigma+2J+\frac{6}{2}\Big)\hbar\omega. \quad (6)$$

The oscillator length is $a_{ho}=\sqrt{\hbar/M\omega}$, and $L_J^{\sigma+2}(\rho^2/a_{ho}^2)$ is the associated Laguerre polynomial. The number of bosons obeys the relation $N_b=\sigma+2J$ with J to classify the representations for a specific value of N_b.

4 Algebraic Correspondences between the IBM and Systems of Cold Atoms

Werner and Castin [18] introduced mappings between zero-energy states and trapped states widely used in cold atoms. A trapped state is merely a quantized state of a harmonic oscillator. One opens the walls of the trap by reaching the zero-frequency limit $\omega=0$ in the harmonic oscillator and obtains the zero-energy state. Werner-Castin mappings were introduced in the solutions of the Schrödinger equation for N trapped cold atoms or particles in general. They preserve the unitary limit and are realized through the generators of the $SO(2,1)$ group in an isomorphic realization to the generators of the conformal group in one dimension - time. An algebraic correspondence for those mappings is established with the simplest form of the $O(6)$ limit of the IBM [15]. In other words, the Werner-Castin mappings [18] correspond to certain relations in the $O(6)$ limit of the IBM under the appropriate algebraic replacements. By this process, one introduces the one-dimensional conformal group in the IBM and writes down the wavefunction for the corresponding zero-energy state which contains a scaling exponent. The scaling exponent arises out of the invariant quantity of dilatation (scale) transformations of the boson number radius ρ. The algebraic correspondence determines that invariant quantity to be the $O(6)$ quantum number σ and the scaling exponent to be the boson number $N_b = \sigma + 2J$ [15]. Table 1 summarizes the main replacements/relations of this correspondence.

Table 1. Algebraic correspondences between the Schrödinger equation of $N = 2$ trapped atoms with the IBM O(6) limit. $r_{1,2}$ is each atom's radial distance from the trap's center, and $l_{1,2}$ are their angular momenta. $L_\pm$, L_0 are the $SO(2,1)$ generators [15].

	$N = 2$ hyperspherical	$O(6)$ IBM
radial variable	$R = \sqrt{r_1^2 + r_2^2}$	boson number radius ρ
dilatation eigenvalue	$\lambda = l_1 + l_2$	quantum number σ
energy	$(\lambda + 2q + 6/2)\hbar\omega$	$(N_b + 6/2)\hbar\omega$, $N_b = \sigma + 2J$
zero energy state	$\psi^0_\lambda = R^{\lambda+2q}$	$\psi^0_\sigma = \rho^{N_b}$
Werner-Castin mapping	$\lvert F^q_\lambda\rangle = L^q_+ e^{-R^2/2a^2_{\hbar o}} \lvert\psi^0_\lambda\rangle$	$\lvert F^J_\sigma\rangle = L^J_+ e^{-\rho^2/2a^2_{\hbar o}} \lvert\psi^0_\sigma\rangle$

By this correspondence one introduces the $SO(2,1)$ group in the Schrödinger equation (5). This group is isomorphic to the conformal group in one dimension [19] in which one deals with three generators that obey the commutation relations

$$[H, D] = -2iH, \quad [K, D] = 2iK, \quad [K, H] = i\hbar^2\omega^2 D. \tag{7}$$

These are the free space Hamiltonian H, the dilatation operator D, and the special conformal operator K and read

$$\begin{aligned} H &= \sum_{i=0}^{5} -\frac{\hbar^2}{2M}\partial_i^2, \quad K = \sum_{i=0}^{5} \frac{1}{2} M\omega^2 q_i^2, \\ D &= \sum_{j=0}^{5} \frac{1}{2i}(\partial_j q_j + q_j \partial_j) = \frac{6}{2i} - i\rho\partial_\rho. \end{aligned} \tag{8}$$

The commutation relations of the $SO(2,1)$ group are closed by three generators L_1, L_2, L_0,

$$2L_1 = \frac{1}{\hbar\omega}(H - K), \quad 2L_2 = D, \quad 2L_0 = \frac{1}{\hbar\omega}(H + K). \tag{9}$$

Ladder operators are defined by the relation $L_\pm = 2(L_1 \pm iL_2)$ which gives

$$L_\pm = \pm iD + \frac{1}{\hbar\omega}(H - K), \tag{10}$$

and in terms of bosons read

$$L_+ = -(\mathbf{d}^\dagger \mathbf{d}^\dagger + s^\dagger s^\dagger), \quad L_- = -(\mathbf{d}\mathbf{d} + ss). \tag{11}$$

These ladder operators create the Werner-Castin mappings in the $O(6)$ limit of the IBM.

5 $A + 2n$ Compound Nucleus at Low Temperatures Like a Cold and Dilute Atomic Gas

One examines the scattering of two slow neutrons ($2n$) with the ground state of a heavy even-even target nucleus (A). The target nucleus is amenable to the $O(6)$ limit of the IBM and the two neutron separation energy S_{2n} determines the length scale of the scattering. In other words, the boson number radius ρ is measured in units of the harmonic oscillator length $a_{ho} = \hbar/\sqrt{MS_{2n}}$ where M is the neutron mass. In that case, one writes down the radial distance between the $2n$ and the IBM state in the form of $R - \rho \equiv r$ and the corresponding wavenumber as k_r.

Channel states are restricted to scalar angular momentum couplings of the form $\Psi(r, \rho) = \sum_n \Psi_n(r)\Phi_n(\rho)$ where $\Psi_n(r)$ is the $2n$ wavefunction, and $\Phi_n(\rho)$ is the IBM wavefunction. The scattering occurs at the cold limit, i.e., at a much lower kinetic energy of the $2n$ concerning the target's S_{2n}. The open channel is the $n = 0$, where the target contains N_b bosons, while closed channel states with the target in higher boson numbers $N_b + 1, 2 \ldots$ are those for $n > 0$. These channel states form the Hilbert space of the IBM-compound Hamiltonian

$$H_c = H(r) + H(\rho) + H(\rho, r). \tag{12}$$

$H(r)$ is the Hamiltonian for the relative kinetic energy of the $2n$ with respect to the target A and the interaction between the two neutrons. $H(\rho)$ is the IBM Hamiltonian for the target, and $H(\rho, r)$ is the $2n$-IBM state interaction term. To investigate the unitary limit of H_c, the interaction terms are specified in analogy with the interactions and the external magnetic field governing the vicinity of Feshbach resonances in cold and dilute atomic gases. There, when the formation of diatomic molecules reaches the unitary limit, the atom-atom unitary interaction induces a molecule-molecule unitary interaction [20]. In H_c, that means one introduces a unitary interaction for the incident neutrons themselves (atom-atom) plus a unitary interaction for the $2n$-IBM state coupling (molecule-molecule). The dilute character of the target's valence space concerning the short range of the strong interaction and the cold $2n$ rationalizes the analogy. Accordingly, a $2n$-IBM state scattering length a_r is introduced through the corresponding effective range expansion as shown in Table 2.

One has the neutron-neutron scattering length a and the pair-IBM state scattering length a_r and the corresponding unitary interactions are

$$\begin{aligned} &\frac{4\pi a\hbar^2}{M}\delta(r_1-r_2) \to \lim_{r_1\to r_2}\Psi_0(r) = \frac{C}{r_1-r_2} - \frac{1}{a}, \\ &\frac{4\pi^3 a_r\hbar^2}{M}\delta(r) \to \lim_{r\to 0}\Psi_0(r,\rho) = \Phi_0(\rho)\Big(\frac{C}{r^4} - \frac{1}{a_r^4}\Big). \end{aligned} \tag{13}$$

These boundary conditions apply to the $2n$ scattering wavefunction $\Psi_0(r)$, and to the channel wavefunction $\Psi_0(r,\rho)$. They replace the two unitary interactions, respectively. The full IBM-compound Hamiltonian now reads

$$H_c = -\frac{\hbar^2}{2M}\Big(\frac{1}{r^5}\frac{\partial}{\partial r}r^5\frac{\partial}{\partial r} - \frac{\lambda(\lambda+4)}{r^2}\Big) + H(\rho) + s^\dagger + s. \tag{14}$$

The $s^\dagger + s$ term changes target states by one s boson. It is the analog of the magnetic field tuning to achieve the open-closed channel crossing. The effective range of the $2n$-IBM state interaction is the r^* as seen in Table 2. That range is determined experimentally by the width of the resonance that corresponds to the channels' crossing through the relation $\Gamma_m = \hbar^2 k_r/Mr^*$.

Intermediate states of the Feshbach formalism are *stationary* states of the $A+2n$ compound nucleus formed by the target plus two neutrons. They serve as resonance states of energy ϵ_n with respect to the total energy E of the open channel. The coupled channels equations read

$$\begin{aligned} (E - H_{PP})P|\Psi\rangle &= H_{PQ}Q|\Psi\rangle, \\ (E - H_{QQ})Q|\Psi\rangle &= H_{QP}P|\Psi\rangle. \end{aligned} \tag{15}$$

Table 2. The solutions of the open channel of the $2n$-A scattering compared to those of $1n$-A scattering. $r_{1,2}$ is the radial distance of each neutron with respect to the heavy A core.

scattering	$1n$-A	$2n$-A
radial variable	r_1	$r = R - \rho$, $R^2 = r_1^2 + r_2^2$
wavenumber	k_1	k_r
scattering wavefunction	$\Psi_0(r_1) = \frac{e^{-ik_1r_1}}{r_1} - S_0\frac{e^{ik_1r_1}}{r_1}$	$\Psi_0(r) = \frac{e^{-ik_rr}}{r^{5/2}} - S_0\frac{e^{ik_rr}}{r^{5/2}}$
effective range expansion	$\frac{1}{a_1(k_1)} = \frac{1}{a_1} - \frac{1}{2}k_1^2r_1^* + \cdots$	$\frac{1}{a_r(k_r)} = \frac{1}{a_r} - \frac{1}{2}k_r^2r^* + \cdots$
cross - section	$\sigma = \frac{4\pi}{k^2+1/a_1^2(k)}$	$\sigma = \frac{(4\pi)^3}{k_r^2+1/a_r^2(k)}$

The projection operators are the open channel $P = |\Phi_0(\rho)\rangle\langle\Phi_0(\rho)|$ and the set of closed channels $Q = \sum_{n>0} |\Phi_n(\rho)\rangle\langle\Phi_n(\rho)|$. Open-open H_{PP} and closed-closed H_{QQ} channel couplings are the unitary interactions and are included in the boundary conditions (13). One examines the coupling of the open channel ($n = 0$) of N_b bosons with the first closed channel ($n = 1$) of $N_b + 1$ bosons. The coupling H_{PQ} now is $H_{10} = \langle N_b + 1|s^\dagger + s|N_b\rangle = \sqrt{N_b + 1}$ and the reverse H_{QP} is the same $H_{01} = \langle N_b|s + s^\dagger|N_b + 1\rangle = \sqrt{N_b + 1}$.

6 Results

The energy scale is normalized to the energy of the target's ground state, i.e., to N_b bosons. The $2n$ s-wave ($\lambda = 0$) open channel solutions are presented in sufficient detail in [15]. Table 2 summarizes the results for the $2n - A$ scattering compared to $1n - A$ scattering. The main difference is the cubic power in the solid angle factor of the cross-section and on the content of the wave numbers.

In absence of a resonance with the intermediate state, the s-wave phase shifts of the $2n$-IBM state scattering give the scattering matrix element $S_0 = e^{2i\delta(k_r)} = e^{-2ik_r a_r}$. Now, one focuses on the first closed channel where the energy of the target (IBM state) is denoted by the capital $E_1 = (N_b + 1 + 6/2)\hbar\omega$ and differs by S_{2n} from the energy of the target in the open channel of N_b bosons. The intermediate state $\Psi_1(r)$ of the $2n$ on that closed channel has, in general, an unknown energy denoted by ϵ_1. Its Schrödinger equation is obtained by the second equation of (15) by setting $H_{10} = 0$. Then, the total energy E is restricted to the energy ϵ_1 of the intermediate state, and its equation reads

$$(T_r + E_1)\Psi_1(r) = \epsilon_1\Psi_1(r). \tag{16}$$

This equation supports a zero-energy state under the condition $E_1 = \epsilon_1$. That condition is satisfied when the energy of the intermediate state of the $2n$ in the compound $A+2n$ nucleus coincides with the bound IBM state of N_b+1 bosons. In other words, the unitarity condition is satisfied when the incident 2n are captured as one s boson. Unitarity manifests itself for that state by turning to the effect on the scattering matrix element. The new element of the scattering matrix reads $S_0' = S_0 S_R$, with the fluctuating term $S_R = 1 - i\Gamma_1/(E - E_1 + i\Gamma_1/2)$. Like in Eq. (3), that fluctuation generates an effective $2n$-IBM state scattering length $a_{r\text{eff}}$ in the same sense with the emergence of the a_{eff} in cold atoms. Therefore, at resonance, the condition $1/a_r(k_r) = 0$ applies and the $a_{r\text{eff}}$ affects the $1/a_r$ part of the effective range expansion. That resonance is measurable through the fluctuation S_R, which generates the compound-elastic cross-section

$$\sigma_{ce} = \frac{(4\pi)^3}{k_r^2}\frac{\Gamma_1^2}{(E - E_1)^2 + (\Gamma_1)^2/4}. \tag{17}$$

The exhaustion of the unitarity bound occurs when the resonance's energy is the two neutron separation energy $E = E_1 = S_{2n}$. The vicinity of the unitary limit is quantified by the width $\Gamma_1 = b_1^2(4M/\hbar^2)k_r$, with $b_1^2 = (N_b +$

$1)|\int dr\Psi_1(r)\Psi_0(r)|^2$ [15]. The latter depends on the neutron mass, the kinetic energy, and the boson number of the closed channel times a spectroscopic factor for the intermediate state.

6.1 Representations of the one-dimensional conformal group in intermediate states

By the action of the $SO(2,1)$ generators to the target states one obtains a whole tower of states

$$(H+K)L_+^k|\Phi_n(\rho)\rangle = (E_n + 2k\hbar\omega)L_+^k|\Phi_n(\rho)\rangle. \tag{18}$$

One solution for the amplitude of the $2n$, $\Psi_1(r)$ in the first closed channel corresponds to $L_+|\Phi_1(\rho)\rangle$ as well as to $L_+^2|\Phi_1(\rho)\rangle$ and consequently to the k-th member of the tower. The intermediate state is coupled with a tower of equally spaced states. Tower states are time-depended [21] emerging from the application of the one-dimensional conformal transformation

$$\tau(t) = \int_0^t \frac{dt'}{\lambda^2(t')}, \quad \tilde{\rho} = \frac{\rho}{\lambda(t)}, \quad \lambda(t) = \sqrt{1+\frac{E^2}{\hbar^2}t^2}, \tag{19}$$

to the time dependent target state $\Phi_1(\rho,t)$

$$\Phi_1(\rho,t) = \Big(e^{-iEt/\hbar} - \epsilon e^{-i(E+2\hbar\omega)t/\hbar}L_+ + \epsilon^* e^{-i(E-2\hbar\omega)t/\hbar}L_-\Big)\Phi_1(\rho,0). \tag{20}$$

The time dependent scale factor $\lambda(t)$ signals the incident wave of $2n$ that perturbs the boson number radius $\rho\to\tilde{\rho}$. Like in cold atoms, infinitesimal changes in the scale factor $\lambda(t) = 1+\delta\lambda(t)$, $\delta\lambda(t) \ll 1$, cause an oscillation of the boson number radius $\rho(t) = (1+\delta\lambda(t))\rho(0)$. By choosing the small parameter $\epsilon = E(2_1^+)/S_{2n}$, the boson number radius oscillates due to the factor [18]

$$\delta\lambda(t) = \epsilon e^{-2i(S_{2n}/\hbar)t} + \epsilon e^{2i(S_{2n}/\hbar)t} + O(\epsilon^2). \tag{21}$$

This is an example of how the modes of energy $E_n \pm 2kS_{2n}$ couple with the intermediate states of the $A+2n$ compound nucleus and span the representations of the one-dimensional conformal group.

7 Conclusions

This contribution briefly reviewed the first exploration of regularity patterns in intermediate states of $A+2n$ compound nuclei [15]. The fluctuations of the cross sections are the experimental means through which those regularities may be tested. One should distinguish those states from doorway states in $A+1n$ compound nuclei that exemplify the tremendous complexity of nuclear excitation spectra in the continuum [5]. Doorway states emerge from a very restrictive coupling term in the continuum Shell Model. On the other hand, intermediate states of $A + 2n$ compound nuclei at unitarity arise out of the simple boson coupling term $s^\dagger + s$. The regularity patterns under discussion are built on the top of those boson states in the continuum and represent the one-dimensional conformal group.

This work did not examine the conditions under which the particular capture of the $2n$ as one boson occurs. Instead, this is the result of the investigation of unitarity in the $A + 2n$ compound nucleus, i.e., that the capture of the slowly incident $2n$ as one boson maximizes the $2n$-IBM state scattering length a_r. The accompanied phenomenological insight is sufficiently important. Namely, the energies and the widths of the fluctuations of the cross-sections of $A + 2n$ compound nuclei propose an experimental case study to examine the unitary limit in nuclear physics.

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