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Radioactive Decay with a Screened Electrostatic Interaction

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Abstract. A phenomenological model for the particle emission half-lives is proposed based on the WKB theory applied to a generalized electrostatic interaction with the inclusion of a screening effect. The screening is modeled through the analytical form of the Hulthen potential. With few approximations, the model can be reduced to a simple empirical formula which recovers the universal decay law when screening is absent. Numerical applications are discussed for the assertion of the impact of the screening effect on the reproduction of data for α and proton emission.

1 Introduction

The proton emission and α decay theoretical studies are based on the simple premise of a one-dimensional barrier tunneling problem [1]. The central observables governing this mechanism are the decay energy and the associated half-life. The later is related to the penetration probability, usually calculated in a semiclassical formalism. The modeling of the potential barrier to be penetrated is the focus of the theoretical studies. Most of these theoretical approaches are concerned with the inner part of the potential barrier which bears nuclear structure information, the outer post-scission part being considered as well understood in terms of the Coulomb electrostatic repulsion. The deviation from this idealized picture is investigated here by considering a Hulthen [2, 3] potential, which generalizes the Coulomb potential by means of a screening effect. This is realized by determining analytically the WKB penetration probability for a Hulthen potential with and without centrifugal contribution, and some approximations of it. As in the case of the Coulomb interaction, such formulas provide useful correlations relating half-lives, decay energy and nucleon numbers, which can be used for the description of experimental data and predictions. Numerical applications of this model in its various forms are recounted for both proton emission and alpha decay phenomena.

2 Theoretical Formulation

From the quantum mechanical point of view, spontaneous emission of a light cluster or particle can be modeled as tunneling through a potential barrier. Here, one separates the barrier into inner and outer regions in terms of the distance between the radii of the daughter nucleus and of the emitted particle. The first region contains information about the formation or the emergence of the particle on the surface of the compound parent nucleus, which leads to the touching configuration. It is defined by the interval between the radius of the parent nucleus R_0 and the distance of the touching configuration $R_t = R_1 + R_p$, where R_1 and R_p are the radii of the daughter nucleus and of the emitted particle, respectively. For simplicity, we consider here that the hard radii are defined as $R_i = 1.2A_i^{1/3}$. There are various ways to describe this part of the potential barrier and its contribution to the emission probability. Here we are concerned with the outer part of the potential barrier, which is traditionally defined as a superposition of a centrifugal energy term

$$V_l(r) = \frac{\hbar^2 l(l+1)}{2\mu r^2} \tag{1}$$

and a repulsive electrostatic potential. $\mu = A_1 A_p/(A_1 + A_p)$ is the reduced mass of the decaying nuclear system defined by the mass numbers of the daughter nucleus A_1 and of the emitted particle A_p. The orbital momentum l of the emitted particle must satisfy the angular momentum and parity conservation laws concerning the initial and final nuclear states. Usually, the Coulomb interaction is the default choice for the electrostatic potential:

$$V_{\mathrm{C}}(r) = \frac{Z_1 Z_p e_p^2}{r}, \tag{2}$$

where Z_1 and Z_p are the daughter and particle charge numbers and e_p being the elementary charge. Ref. [4] for the first time considered a different approach for this interaction, namely employing a Hulthen [2,3] type potential,

$$V_{\mathrm{H}}(r) = \frac{\delta e_p^2 Z_1 Z_p}{e^{\delta r} - 1}. \tag{3}$$

This potential is a generalization of the Coulomb potential with a screening effect defined by the parameter δ. It has a short range which is adjustable, whereas Coulomb potential is of long range. Therefore, the Hulthen potential allows us to modify the usual Coulomb interaction by means of its convergence range δ which is considered to include here the various unaccounted secondary contributions of the finite size effects, surface tension, surface diffuseness, nuclear interaction, deformation and vibration of nuclear shapes, deviations from the electrostatic approximation, electrodynamics, inhomogeneous charge distribution, etc. Let us now discuss the consequences of this potential on the decay half-live of the parent nucleus, which can be generically defined as

$$T_{1/2} = \frac{\ln 2}{P}, \tag{4}$$

where P is the probability of the proton to penetrate a phenomenological potential barrier. It can be divided in two parts $P = P_{\text{in}}P_{\text{out}}$: one associated with the inner potential which can include for example also a contribution from the assault frequency on the barrier or nuclear structure aspects, and a second part dealing precisely with barrier penetration defined by the outer potential $V_{\text{out}} = V_l(r) + V_{\text{H}}(r)$. The later is calculated by means of the WKB approximation:

$$P_{out} = \exp\left\{-\frac{2}{\hbar}\int_{R_t}^{R_{\text{out}}} \sqrt{2\mu\left[V_{\text{out}}(r) - Q_p\right]dr}\right\}, \tag{5}$$

where R_{out} is the second turning point defined by the total decay energy $Q_p = V_{\text{out}}(R_{\text{out}})$.

The calculation of the integral $I = -\hbar \ln P_{\text{out}}/(2\sqrt{2\mu})$ is preceded by the Langer transform demanded in the WKB procedure for spherically symmetric systems. It amounts to the change of $l(l+1)$ with $(l+1/2)^2$ [5]. Although I can be easily calculated numerically for specific nuclei, it is convenient to have its analytical estimation in order to track various dependencies. This can be achieved by considering the following approximation for the centrifugal energy [6]:

$$\frac{1}{r^2} \approx \frac{\delta^2}{\left(e^{\delta r} - 1\right)^2}, \tag{6}$$

which is valid for small values of δ screening. In this approximation, the $Q_p = V_{\text{out}}(R_{\text{out}})$ equation provides the exit radius

$$R_{\text{out}} = \frac{1}{\delta}\log\left[\frac{2V_1}{\sqrt{V_0^2 + 4V_1 Q} - V_0} + 1\right], \tag{7}$$

where

$$V_0 = \delta e_p^2 Z_1 Z_p, \qquad V_1 = \frac{\delta^2\hbar^2\left(l+\frac{1}{2}\right)^2}{2\mu}. \tag{8}$$

Through successive algebraic manipulations, the integral I can be written as

$$I = \frac{1}{\delta}\left[I_1(r) + I_2(r)\right]\Big|_{R_t}^{R_{\text{out}}}, \tag{9}$$

with the two terms having the following expressions:

$$I_1(r) = -\sqrt{V_1x^2 + V_0x - Q_p} + \sqrt{Q_p}\arcsin\left[\frac{xV_0 - 2Q_p}{x\sqrt{4Q_pV_1 + V_0^2}}\right] - \frac{V_0}{2\sqrt{V_1}}\log\left[2\sqrt{V_1\left(V_1x^2 + V_0x - Q_p\right)} + V_0 + 2V_1x\right], \tag{10}$$

$$I_2(r) = \sqrt{V_1 y^2 + U_0 y - U_1} - \sqrt{U_1}\arctan\Big[\frac{yU_0 - 2U_1}{2\sqrt{U_1\,(V_1 y^2 + U_0 y - U_1)}}\Big] + \frac{U_0}{2\sqrt{V_1}}\log\Big[2\sqrt{V_1\,(V_1 y^2 + U_0 y - U_1)} + U_0 + 2V_1 y\Big], \tag{11}$$

where new variables

$$x = (e^{\delta r} - 1)^{-1}\,, \qquad y = 1 + (e^{\delta r} - 1)^{-1}, \tag{12}$$

are used together with the following notations:

$$U_0 = V_0 - 2V_1\,, \qquad U_1 = Q_p + V_0 - V_1. \tag{13}$$

The centrifugal potential can be extracted from the WKB integral or omitted altogether. The later option is available for the alpha decay, whose mass is four times larger than that of the single emitted proton. The WKB estimation of penetrability for a pure Hulthen potential then will have a different exit radius:

$$R^{\mathrm{H}}_{\mathrm{out}} = \frac{1}{\delta}\ln\left(\frac{\delta V_0}{Q} + 1\right). \tag{14}$$

The WKB integral just for the Hulthen potential then becomes

$$\begin{aligned} I^{\mathrm{H}} &= \int_{R_t}^{R_{\mathrm{out}}} \sqrt{\frac{\delta V_0}{e^{\delta r} - 1} - Q_p dr} \\ &= \frac{2}{\delta}\left[\sqrt{Q_p + \delta V_0}\arctan\sqrt{\frac{\frac{\delta V_0}{e^{\delta R_t} - 1} - Q_p}{Q_p + \delta V_0}} \right. \\ &\qquad \left. - \sqrt{Q_p}\arctan\sqrt{\frac{\delta V_0}{Q_p(e^{\delta R_t} - 1)} - 1}\right]. \end{aligned} \tag{15}$$

As δ is supposed to have small values, the above expression can be then approximated as a first order expansion in δ:

$$\begin{aligned} I^{\mathrm{H}} \approx &- R_t\sqrt{Q_p}\Big(1 + \frac{\delta V_0}{4Q_p}\Big)\sqrt{\frac{V_0}{Q_p R_t} - 1} \\ &+ \frac{V_0}{\sqrt{Q_p}}\Big(1 - \frac{\delta V_0}{4Q_p}\Big)\arctan\sqrt{\frac{V_0}{Q_p R_t} - 1}. \end{aligned} \tag{16}$$

An additional layer of approximations can be realized based on the condition $V_0/(Q_p R_t) \gg 1$. This implies the validity of the following asymptotic expressions:

$$\sqrt{x-1} \approx \sqrt{x}\,, \qquad \arctan\sqrt{x-1} \approx \frac{\pi}{2}. \tag{17}$$

Plugging these correspondences into Eq.(16), one obtains

$$I=-\sqrt{R_tV_0}\left(1+\frac{\delta V_0}{4Q_p}\right)+\frac{V_0\pi}{2\sqrt{Q_p}}\left(1-\frac{\delta V_0}{4Q_p}\right). \tag{18}$$

Using the above estimation for the penetration integral of the outer barrier, one can write down the following formula for the decimal logarithm of the half-live as [7,8]:

$$\log_{10}T_{1/2}=A\chi\left(1-D\frac{\chi^2}{\eta\rho^2}\right)+B\rho\left(1+D\frac{\chi^2}{\eta\rho^2}\right)+C, \tag{19}$$

where A,B,C and D are adjustable parameters gathering different physical constants. Notably, the free term comes from $C=\log_{10}\left(\hbar\ln 2/P_{\text{in}}\right)$. In the above formula, one used the standard notations

$$\chi=Z_pZ_1\sqrt{\frac{A_pA_1}{(A_p+A_1)Q_p}}, \tag{20}$$

$$\rho=\sqrt{\frac{Z_pZ_1A_pA_1\left(A_p^{1/3}+A_1^{1/3}\right)}{A_p+A_1}}, \tag{21}$$

$$\eta=A_p^{1/3}+A_1^{1/3} \tag{22}$$

of the universal decay law (UDL) [9,10]:

$$\log_{10}T_{1/2}(\text{UDL})=A\chi+B\rho+C, \tag{23}$$

which is recovered form Eq.(19), when the screening $D=\delta e_p^2/4$ reduces to zero. The universality of the UDL is exemplified by the fact that other widespread empirical correlations emerge from it, by employing additional approximations. For example, when the touching radius is rendered constant, that is $R_t\sim\eta\approx const.$, the Ni-Ren-Dong-Xu formula [11] is obtained

$$\log_{10}T_{1/2}(\text{Ni-Ren-Dong-Xu})=AZ_1Z_p\sqrt{\frac{\mu}{Q_p}}+B\sqrt{\mu Z_1Z_p}+C. \tag{24}$$

On the other hand, considering an invariant reduced mass $\mu\approx const.$ and $A_d\gg A_c$, one recovers the Royer formula [12]

$$\log_{10}T_{1/2}(\text{Royer})=A\frac{Z_1}{\sqrt{Q_p}}+B\sqrt{Z_1}A_1^{\frac{1}{6}}+C. \tag{25}$$

In both of the above formulas the parameters A,B,C are distinct, as they incorporate different quantities.

The centrifugal contribution can be reinserted into Eq.(19) in the same manner as in the extension of the UDL for proton emission [13], where this dependence is of a crucial role. Basically, the formula (19) is amended by a $l(L+1)/\rho$ term with an additional adjustable parameter.

Presently, few variations of the modified UDL formula (19) are available. There is an option to consider differentiated contribution of the screening for the first two terms, which implies the use of two parameters $D_1 \neq D_2$ instead of a single parameter D [14]. The same screening correction to the UDL can be made to be deformation dependent as in Ref. [15]. Another more straightforward mode of improvement is by considering additional terms depending for example on isospin [14].

3 Numerical Results

Before discussing the performance of the models involving screening in reproducing experimental data, it is instructive to visualize the effect of the screening on the bare Coulomb interaction, and its effect in various disintegration scenarios. To this purpose, we plotted in Fig.1 the schematic representation of the outer potential for two selected cases of favored proton emission and alpha decay, in the presence and in the absence of the screening. The effect of the screening is obvious in the overall lowering of the barrier height, which leads also to a narrower barrier represented by a shorter exit radius. The latter feature is more significant in the case of proton emission where the decay energy is usually several times smaller than for the alpha decay process.

Various models which stem from a Hulthen potential describing the electrostatic interaction were used to calculate the half-lives of alpha decay [7, 14, 17], proton emission [4, 15] and even two-proton emission [15]. The introduction of the screening greatly improves the agreement with experimental data. The precise form given by the penetration Gamow factor (9), was used in phenomenological models with various descriptions of the inner potential barrier [4, 17]. The analytical formalism of these approaches is found to depend mainly on the screening parameter and was successfully applied for the description of proton emission and alpha decay from heavy and superheavy nuclei, both favored and unfavored. The dependence of the theoretical results, only on the screening measure, is used to identify the values corresponding to each data point. In this way, it was possible to study the systematic evolution of the screening as a function of nucleon numbers and deformation in case of the alpha decay, where there is more data. It was therefore found that the screening is sensible to shell filling, be it for spherical or deformed nuclei. More precisely, it has a discontinuity at magic numbers. For example, the screening will decrease up to the magic neutron number $N = 126$, and then suddenly increases above it and then starts another decreasing cycle. From the correlation of the screening for individual nuclei with their corresponding quadrupole deformation, it was found that the screening parameter increases with deformation and then start to decrease for very large deformations. The later behaviour is in agreement with the shell filling mechanism discussed above, because at higher deformation emerge some substantial shell gaps. A similar correspondence between screening and deformation was discussed for the proton emission data [15] by means of the modified UDL formula (19). In this case, it was found that there is no need for screening when the decaying nuclei are very deformed. As a result, a deformation dependence was introduced for the screening parameter from (19), by means of a step function of quadrupole deformation localized at $\beta_2 = 0.24$.

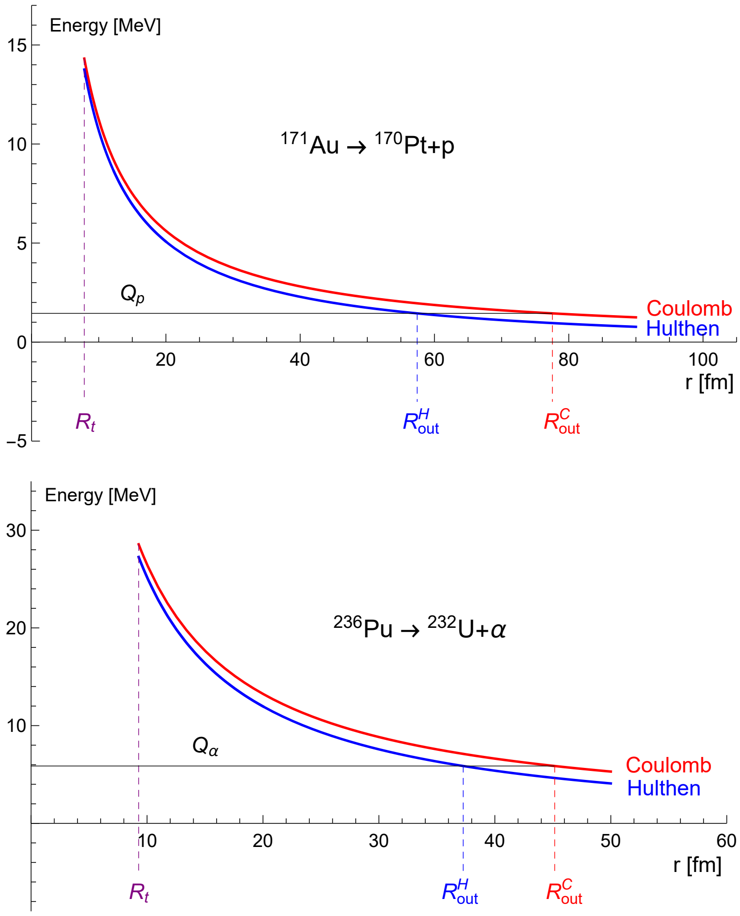

Figure 1. Schematic representation of the outer potential for the favored proton emission from ^{171}Au (top) and for the favored α decay of ^{236}Pu (bottom) as a function of the distance between the centers of the decayed fragments, with screened ($\delta = 0.01$) and bare electrostatic interactions. The experimental values of the decay energies are collected from Ref. [16].

4 Conclusion

An analytical estimation is proposed for the penetrability of a phenomenological barrier defined by a screened electrostatic interaction modeled by a Hulthen potential, with and without a centrifugal contribution. Through a series of approximations, the result can be put into a form of a modified universal decay law. The effect of the screening is discussed in connection to previous numerical applications of the exact formula in the frame of some phenomenological models, or the empirical formula to alpha decay and proton emission experimental data. It is thus concluded that screening, although considered as an averaging effect for data reproduction, is very sensible to shell structure and consequently on deformation of the involved nuclei.

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References

[1] G. Gamow, *Z. Phys.* **51** (1928) 204.
[2] L. Hulthen, *Ark. Mat. Astron. Fys. A* **28** (1942) 52.
[3] L. Hulthen, M. Sugawara, S. Flugge (Eds.), *Handbuch der Physik*, Springer, Berlin, Heidelberg (1957).
[4] R. Budaca, A.I. Budaca, *Eur. Phys. J. A* **53** (2017) 160.
[5] R.E. Langer, *Phys. Rev.* **51** (1937) 669.
[6] E.D. Filho, R.M. Ricotta, *Mod. Phys. Lett. A* **10** (1995) 1613.
[7] A.I. Budaca, *Eur. Phys. J. A* **57** (2021) 41.
[8] A.I. Budaca, *AIP Conf. Proc.* **3181** (2024) 050003.
[9] C. Qi, F.R. Xu, R.J. Liotta, R. Wyss, *Phys. Rev. Lett.* **103** (2009) 072501.
[10] C. Qi, F.R. Xu, R.J. Liotta, R. Wyss, M.Y. Zhang, C. Asawatangtrakuldee, D. Hu, *Phys. Rev. C* **80** (2006) 044326.
[11] D. Ni, Z. Ren, T. Dong, C. Xu, *Phys. Rev. C* **78** (2008) 044310.
[12] G. Royer, *J. Phys. G: Nucl. Part. Phys.* **26** (2000) 1149.
[13] C. Qi, D.S. Delion, R.J. Liotta, R. Wyss, *Phys. Rev. C* **85** (2012) 011303(R).
[14] D.T. Akrawy, A.I. Budaca, G. Saxena, Ali H. Ahmed, *Eur. Phys. J. A* **58** (2022) 145.
[15] R. Budaca, A.I. Budaca, *Nucl. Phys. A* **1017** (2022) 122355.
[16] M. Wang, W.J. Huang, F.G. Kondev, G. Audi, S. Naimi, *Chin. Phys. C* **45** (2021) 030003.
[17] R. Budaca, A.I. Budaca, *Chin. Phys. C* **44** (2020) 124102.