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Non-Radial Oscillation Modes of Twin Stars in the Cowling Approximation

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Abstract. The observational distinction of two compact stars with comparable masses and significantly different radii would be a strong indication of a phase transition in dense nuclear matter. Motivated by previous works that attempted to investigate distinct observable quantities characterizing twin stars (stars with equal mass but different internal composition), in the present work we examine the frequencies of their non-radial oscillation modes. The analysis is performed within the relativistic Cowling approximation and to enrich and extend our study we use two distinct hadronic models attempting to qualitatively estimate the role of the stiffness (of the equation of state) regarding the resulting pulsation frequencies.

1 Introduction

Neutron stars are considered to be unique laboratories for the study of dense nuclear matter [1]. Interestingly, the extreme conditions that characterize their inner core have motivated physicists to explore exotic scenarios regarding the dominant degrees of freedom in their center [2–4]. One of these scenarios involves a first-order phase transition which leads to a deconfined quark matter core covered by a layer of hadrons [2, 4]. In such case, the stellar objects are no longer composed solely of neutrons and protons and therefore the name ”neutron stars” is replaced by the name ”hybrid stars”.

In the past years, a lot of research has been focused on the hadron-quark phase transition hypothesis and the structural properties of hybrid stars [5–11]. In that direction, several studies have investigated signature phenomena that may be associated with the existence of deconfined strange quark matter cores [12–21]. One of these signature predictions is the existence of twin star configurations (or *i.e.* the existence of a third family of compact objects) [12, 13]. In principle, using a hadronic equation of state (EOS) leads to compact star models where each configuration of specific mass is associated with a single radius value. Notably, if one uses an EOS which incorporates a strong first-order phase transition then two stars with equal mass but different radii could theoretically exist.

After the theoretical prediction regarding the possible existence of twin stars several works have attempted to postulate different approaches that could lead to their observational distinction. Lyra *et al.* [22] examined the role of compactness on the cooling of twins, showing that only stars with important differences in their radii have distinct cooling behavior. Tan *et al.* [23] studied signatures that appear in binary Love universal relations when EOSs involving strong phase transitions are considered. Landry and Chakravarti [24] discussed the possibility of distinguishing twin pairs with next-generation GW detectors by measuring their distinct tidal deformabilities. In Ref. [25], the authors have indicated that twin configurations could be characterized by different r mode instability windows and hence that two stars with identical mass, and similar frequency and temperature could behave differently with respect to r mode gravitational wave emission. Finally, in Ref. [26] the authors have indicated that the discovery of gravitational waves from f mode oscillations could potentially allow for the distinction of twin configurations.

In the present conference contribution, we aim to investigate the differences in the f and p_1 mode frequencies of twin stars (in the relativistic Cowling approximation). Our preliminary analysis involves a flexible model for the construction of hybrid EOSs, which allows us to examine the impact of altering the phase transition properties on the resulting frequency deviations. Furthermore, we employ two distinct EOSs for the parametrization of ordinary nuclear matter in order to clarify how the stiffness of the nuclear EOS may alter our findings.

2 Equation of State

For the construction of hybrid EOSs we employ the widely-known constant speed of sound parametrization. In that framework the energy density reads [8,9]

$$\mathcal{E}(P) = \begin{cases} \mathcal{E}_{\rm HADRON}(P), & P \leq P_{\rm tr} \\ \mathcal{E}(P_{\rm tr}) + \Delta\mathcal{E} + (c_s/c)^{-2}(P - P_{\rm tr}), & P > P_{\rm tr}, \end{cases} \tag{1}$$

where $P_{\rm tr}$ denotes the transition pressure, c_s/c is the speed of sound divided by the speed of light and $\Delta\mathcal{E}$ stands for the energy density discontinuity. Notably, the first line of Eq. (1) corresponds to the description of the hadronic phase, while the second line to the quark matter part. It is worth noting that the aforementioned model is motivated by Nambu-Jona-Lasinio models which display nearly constant speed of sound [9]. In the present study, following several related works, we are going to use the maximally stiff parametrization $c_s/c = 1$ [27–39].

Notably, the presence of a density discontinuity does not suffice for the emergence of twin star configurations. In particular, the energy density jump has to exceed a critical value in order for this to occur. Such a critical value has been found in the work of Seidov [40] and it is given by

$$\Delta\mathcal{E}_{\rm cr} = \frac{1}{2}\mathcal{E}_{\rm tr} + \frac{3}{2}P_{\rm tr}. \tag{2}$$

The expression in Eq. (2) sets an approximate benchmark for the energy density jump that leads to the existence of a third family of compact objects.

For the hadronic part of the EOS, we employ two distinct relativistic mean-field models, namely the GRDF-DD2 [41] and the NL3 model [42]. Notably, both EOSs are widely used, with the later being stiffer resulting in compact star models with larger radii and maximum mass.

3 Non Radial Oscillation Modes

In the present work, we study the non-radial oscillation modes of compact objects following the formalism presented in Ref. [14]. In particular, we employ the Cowling approximation [43], which only accounts for the pulsation of the fluid and neglects the perturbation in the metric. In this framework, considering the background metric that describes spherically symmetric objects [14, 15]

$$ds^2 = -e^{2\phi(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3}$$

the system of differential equations that needs to be solved in order to obtain the desired mode frequencies is given by

$$\frac{dW}{dr} = \frac{d\mathcal{E}}{dP}[\omega^2 r^2 e^{\lambda-2\phi}V + \phi' W] - l(l+1)e^{\lambda}V, \tag{4}$$

$$\frac{dV}{dr} = 2\phi' V - e^{\lambda}\frac{W}{r^2}, \tag{5}$$

where W and V define the fluid Lagrangian displacement vector

$$\xi_i = (e^{-\Lambda}W, -V\partial_\theta, -V\sin^{-2}\theta\partial_\phi)r^{-2}Y_{lm}e^{i\omega t}, \tag{6}$$

with Y_{lm} standing for the well-known spherical harmonics. The parameter ω appearing in Eq. (4) corresponds to the mode frequency, while the functions λ and ϕ (defining the background metric) result from the solution of the Tolman-Oppenheimer-Volkov equations.

In order to solve the Eq. (4), (5) Strum-Luville problem, one needs an appropriate set of boundary conditions. In particular, near the center of the star ($r \to 0$) W and V have the form [14]

$$W(r) = Cr^{l+1},\ V(r) = -Cr^l/l, \tag{7}$$

where C is an arbitrary constant, while at the surface of the star ($r = R$) the following condition needs to be met

$$[\omega^2 r^2 e^{\lambda-2\phi}V + \phi' W]_{r=R} = 0. \tag{8}$$

Notably, in the presence of a density discontinuity (due to a phase transition) an additional junction condition is required to solve the aforementioned differential equations. More precisely, at the interface of the discontinuity

$$W_+ = W_- \tag{9}$$

$$V_+ = \frac{e^{2\phi-\lambda}}{\omega^2 R_t^2} \left(\frac{\mathcal{E}_- + P}{\mathcal{E}_+ + P} [\omega^2 R_t^2 e^{\lambda-2\phi} V_- + \phi' W_-] - \phi' W_+ \right), \quad (10)$$

where R_t is the radius at which the discontinuity appears and the signs $+, -$ indicate the function values at each side of the boundary between the two phases [14].

The numerical solution of the above eigenvalue problem results into a spectrum of frequencies associated with different oscillation modes. Each mode is characterized by the number of nodes in the eigenfunctions that are derived. In the present study, we will work (for quadrupole $l = 2$ pulsations) with the two modes of the lowest order, the f and p_1 modes, which are characterized by zero and one node, respectively.

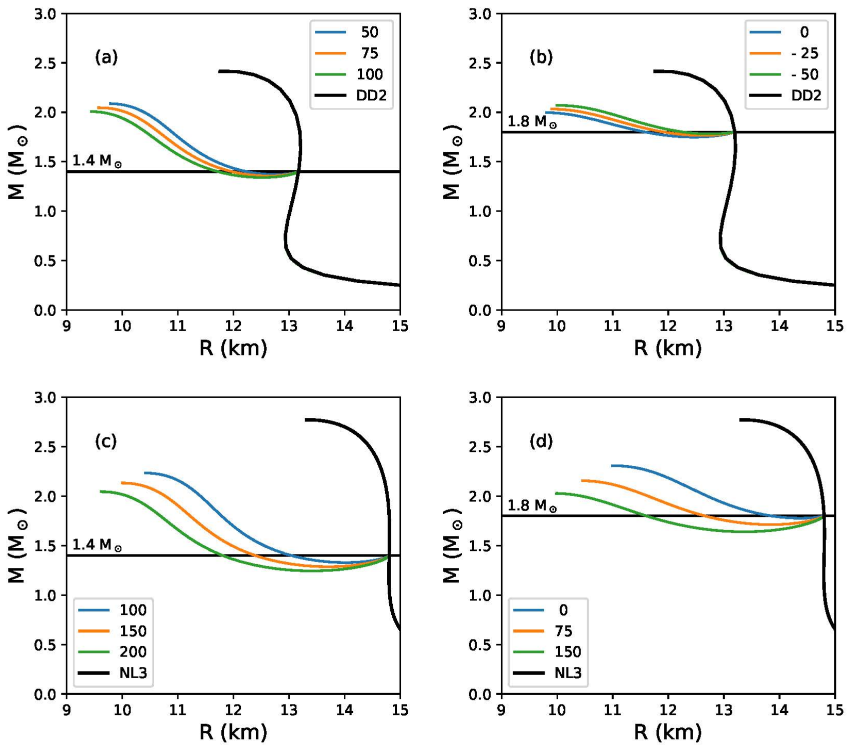

Figure 1. Mass-radius diagrams for the EOSs constructed in the present study. Panels a) and c) contain the results for $1.4M_\odot$ twin stars, while panels b) and d) for $1.8M_\odot$. For panels a) and b) the GRDF-DD2 EOS has been employed, while for panels c) and d) the NL3 model was used. The numbers appearing in the legends denote the difference between the energy density jump and the critical energy density jump (defined by Eq. (2)) for each EOS (units are MeV fm^{-3}).

4 Results and Discussion

In order to study the differences in the non radial oscillation mode frequencies of twin stars we have constructed a large set of hybrid EOSs by following the analysis of Section 2. In Figure 1, one can find the mass-radius dependence predicted by each EOS. The hybrid models are parametrized via their density jump value and transition density. In particular, the numbers appearing in the label of each curve represent the difference between the energy density jump and the critical energy density jump for each of the derived EOSs. The resulting models predict twin stars with 1.4 or $1.8 M_{\odot}$ allowing us to investigate the potential impact of the transition density in the resulting frequency deviations. Finally, all of the chosen parametrizations are compatible to the existence of massive compact

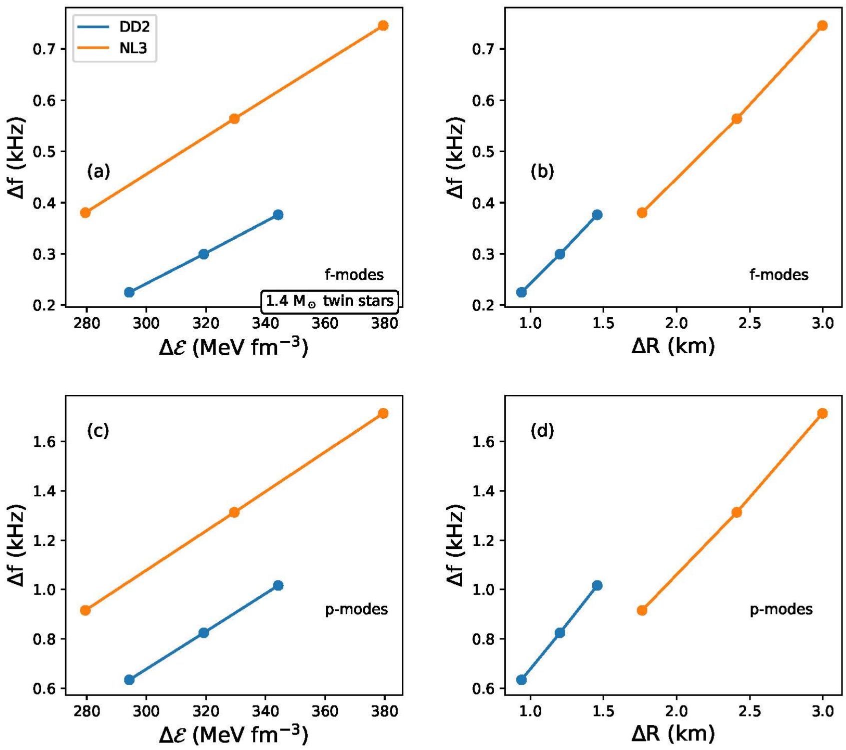

Figure 2. Panel a): The difference in f mode frequencies of 1.4 $M_{\odot}$ twin stars as a function of the energy density jump. Panel b): The difference in f mode frequencies of 1.4 $M_{\odot}$ twin stars as a function of their radius difference. Panel c): The difference in p_1 mode frequencies of 1.4 $M_{\odot}$ twin stars as a function of the energy density jump. Panel d): The difference in p_1 mode frequencies of 1.4 $M_{\odot}$ twin stars as a function of their radius difference.

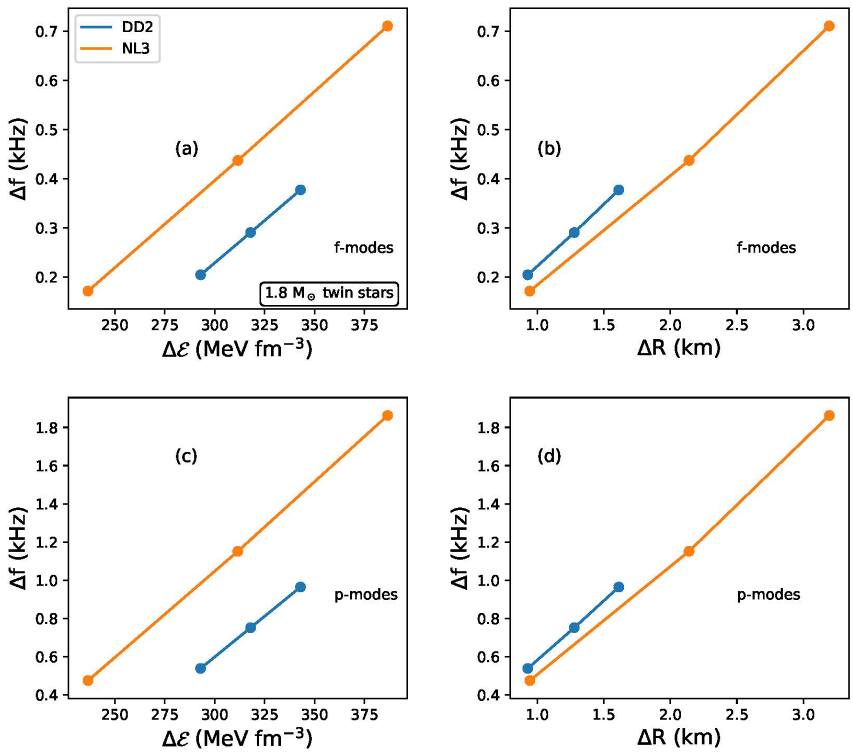

Figure 3. Same as Figure 2 but for 1.8 $M_{\odot}$ twin stars.

stars as the conservative $2M_{\odot}$ maximum mass constraint is fulfilled for every model.

Figure 2(a) illustrates the deviation between the f mode frequencies, while Figure 2(c) between the p_1 mode frequencies, for $1.4M_{\odot}$ twin stars, as a function of the energy density jump. As one can observe, the larger the energy density jump, the larger the difference between the frequencies of twin configurations. Notably, the effect of the EOS stiffness appears to be crucial in our analysis. In particular, neutron stars constructed with the stiffer NL3 EOS are characterized by lower f and p_1 mode frequencies compared to those constructed with the GRDF-DD2 model. However, as shown in Figures 2 (a) and (c) the frequency differences between twins appear to be higher when the NL3 model is employed (for the same energy density jump). We find that in the case of the NL3 model there may be a $\sim 35-40\%$ difference in the f and p_1 mode frequency of twin configurations, while for the GRDF-DD2 model a $\sim 15\%$ deviation. Note that for the specific hadronic models (and mass configurations) one cannot obtain larger frequency deviations, as this would require higher energy density jump values which are excluded by the $2M_{\odot}$ maximum mass constraint.

Figures 2(b) and (d) depict the frequency deviations as function of the radius difference of $1.4M_{\odot}$ twin stars for the f and p_1 modes, respectively. As it is clear, the larger the structural differences of twin stars, the larger the deviations in the frequencies. Such a result is, however, somewhat expected since the larger structural differences are produced by larger energy density jump values.

Another interesting remark is that the dependence of the difference between the f and p_1 mode frequencies on the energy density jump appears to be linear for both hadronic EOSs. While the exact line (describing the dependence) is affected by the selected hadronic model, its slope appears to be nearly identical for both of the employed EOSs. A analogous behavior can also be observed for the dependence between the frequency deviation and the radius difference of twins, but in this case the slopes are not as similar.

Finally, Figure 3 contains the same results with Figure 2, but for $1.8M_{\odot}$ twin star configurations. Interestingly, we find that the consideration of a different transition pressure does not appear to have important impact on the resulting f and p_1 mode frequency deviations of twin pairs.

5 Summary

In this conference contribution we have presented some first results on the f and p_1 mode frequency deviations of twin stars by utilizing the relativistic Cowling approximation. We saw that the stiffness of the hadronic EOS appears to play a crucial role in our analysis, as the stiffer model was found to be connected to significantly larger frequency differences. In addition, the width of the energy density discontinuity was found to be positively correlated with the frequency deviations. With regard to the impact of the transition density, we found that its variation has only minimal impact on our findings.

This preliminary study could be extended in various directions. Firstly, while the Cowling approximation may be sufficient for a qualitative analysis, we aim to solve the fully relativistic problem, which will provide greater accuracy to the obtained frequencies. Secondly, an interesting subject would be to also vary the stiffness in the quark matter part of the EOS in an attempt to include its effects in our investigation. In any case, refining the study on the oscillation modes of twin stars in combination with future observations may hopefully provide important insight regarding the nature of the central region of compact objects.

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